MATH 145 Final Exam 16 December 2013

Name: ____

Do any 6 of the following problems.

Please circle the problems you want to be graded.
All statements must be proved. Answers without proofs may receive 0 credit.

1. Three corner squares are removed from a 9×9 board. Prove that the remaining board cannot be covered with dominoes.
2. Ten points are chosen randomly in a 6×6 square. Prove that at least two of these points are within distance 3 of each other.
3. Prove that for any natural number n, the number $n^3 + 5n$ is divisible by 6.
4. Two players play the following game:
   - turns alternate;
   - at each turn, a player removes 1, 2, or 3 counters from a pile that initially contains 123 counters;
   - the player who removes the last counter loses.

   Find a winning strategy for one of the players (clearly identify whether you have to go first or second in order to win, how you will play throughout the game, and show that you will always be able to move as you described).
5. Find an equation of the line with a negative slope and passing through the point $(1, 1)$ such that the triangle bounded by this line and the axes is divided by the parabola $y = x^2$ into two regions of equal area.
6. We start with the number 2^{2013}. We compute the sum of its digits, and then the sum of the digits of the result, and then again and again until only one digit remains. What is it?
7. A map of the river and the bridges in Konigsberg is given (see next page). As we know from a problem in the book, it is not possible to design a tour of the town that crosses each bridge exactly once and returns to the starting point. Could the citizens of Konigsberg create such a tour by building one new bridge?
8. Find the 2013th derivative of $f(x) = \sin(x)$.

For extra credit:

- Suppose we are given ten planes in a general position (i.e. no two are parallel, no three are parallel to the same line, no four have a common point). Into how many (3-dimensional) regions do they divide $\mathbb{R}^3$?