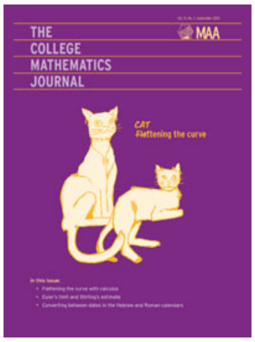

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Problems and Solutions

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PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than March 15, 2021. Sending both pdf and TEXfiles is ideal.

PROBLEMS

Problem 1177, published in the May 2020 issue, was a duplicate of problem 1163 from the November 2019 issue. Because of this, solutions to problem 1177 received before the solutions column for November 2020 issue is prepared (approximately the end of June) will be treated as solutions to problem 1163. Solutions to problem 1177 received after that time will not be able to be used. We apologize for the error.

1181. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $k > 0$ be a real number. Calculate the following:

1. $L := \lim_{n\to\infty} \int_0^1 \left(\frac{\sqrt[n]{x}+k-1}{k}\right)^n dx$, and
2. $\lim_{n\to\infty} n\left(\int_0^1 \left(\frac{\sqrt[n]{x}+k-1}{k}\right)^n dx - L\right)$.

1182. *Proposed by Adam Hammett, Cedarville University, Cedarville, OH.*

Let $c \in \mathbb{R}$, let $\{a_k\}_{k\geq 1}$ be a sequence of real numbers satisfying $a_k - a_{k-1} \geq a_{k+1} - a_k \geq 0$ for all $k \geq 2$, and introduce the power series

$$\chi(c, \{a_k\}, x) := \sum_{n\geq 2}(a_{n-1} - c)\frac{(-1)^n}{x^n}.$$

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1. Find a real number $r > 0$ such that $\chi(c, \{a_k\}, x)$ converges absolutely for $x > r$ and all choices of c and $\{a_k\}$, but $\chi(c, \{a_k\}, r)$ diverges for some choice of c or $\{a_k\}$, and
2. Prove that there exists a function $f(c, \{a_k\}) \geq r$ and a threshold value c^* such that $\chi(c, \{a_k\}, x) > 0$ for each $c < c^*$ and $x > f(c, \{a_k\})$, and $\chi(c, \{a_k\}, x) < 0$ for each $c > c^*$ and $x > f(c, \{a_k\})$. Give an explicit formula for $f(c, \{a_k\})$ and value for c^*.

1183. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.*

Let n be an odd positive integer. Suppose that the integers $1, 2, \ldots, 2n$ are placed around a circle in arbitrary order.

1. Show that there exist n of these numbers, placed in successive locations around the circle, having sum S_1 satisfying $S_1 \geq n^2 + \frac{n+1}{2}$,
2. Show that there exist n of these numbers, placed in successive locations around the circle, having sum S_2 satisfying $S_2 \leq n^2 + \frac{n-1}{2}$, and
3. Show that it is possible to place the $2n$ numbers around the circle in such a way that the sum of every n of these numbers, placed in successive locations around the circle, has sum S_3 satisfying $n^2 + \frac{n-1}{2} \leq S \leq n^2 + \frac{n+1}{2}$.

1184. *Proposed by Seán Stewart, Bomaderry, NSW, Australia.*

Evaluate the following integral:

$$\int_0^\infty \int_0^\infty \frac{\sin x \sin(x+y)}{x(x+y)} dx dy.$$

1185. *Proposed by Greg Oman, University of Colorado, Colorado Springs, Colorado Springs, CO.*

Suppose that S is a commutative ring with identity 1. A subring R of S is called *unital* if $1 \in R$. For the purposes of this problem, call S *special* if S has the following properties:

(a) S has a proper unital subring,
(b) there exists a prime ideal of S which is not maximal, and
(c) if R is any proper unital subring of S, then every prime ideal of R is maximal.

Prove the existence of a special ring or show that no such ring exists.

Convergent or divergent?

1156. *Proposed by George Stoica, University of New Brunswick, New Brunswick, Canada.*

Consider a sequence $(x_n)_{n\geq 1}$ of real numbers which satisfies $\frac{x_{n+1}}{x_n} \to 1$ as $n \to \infty$ and $a \leq \frac{x_{2n}}{x_n} \leq b$ for some positive real numbers a and b and sufficiently large n. Prove the following:

(a) $\sum_{n=1}^{\infty} x_n < \infty$ if $b < \frac{1}{2}$, and
(b) $\sum_{n=1}^{\infty} x_n = \infty$ if $a > \frac{1}{2}$.

Solution by Missouri State Problem Solving Group.

Since $\frac{x_{n+1}}{x_n} > 0$ for sufficiently large n, we see x_n will eventually have constant sign. However, (a) is trivial and (b) is impossible for negative x_k, hence we assume each $x_k > 0$.

(a) Let $b < 1/2$. It follows that $\frac{1}{2b} > 1$, and therefore there exists c such that $1 < c < \frac{1}{2b}$. Since $\frac{x_{n+1}}{x_n} \to 1$, we have $x_{n+1} < c\, x_n$ for sufficiently large n. Hence, with no loss of generality, we may also assume

$$x_{n+1} < c\, x_n \ \text{ and } \ x_{2n} \leq b\, x_n \text{ for all } n.$$

Note that for $r = b(1+c)$,

$$0 < r < b\left(1 + \frac{1}{2b}\right) = b + 1/2 < 1.$$

For each $n \geq 0$ define

$$y_n = x_{2^n} + x_{2^n+1} + x_{2^n+2} + \cdots + x_{2^{n+1}-1}.$$

Then

$$\begin{aligned} y_{n+1} &= (x_{2^{n+1}} + x_{2^{n+1}+1}) + (x_{2^{n+1}+2} + x_{2^{n+1}+3}) + \cdots + (x_{2^{n+2}-2} + x_{2^{n+2}-1}) \\ &< (1+c)(x_{2^{n+1}} + x_{2^{n+1}+2} + x_{2^{n+1}+4} + \cdots + x_{2^{n+2}-2}) \\ &\leq b(1+c)(x_{2^n} + x_{2^n+1} + x_{2^n+2} + \cdots + x_{2^{n+1}-1}) \\ &= r\, y_n. \end{aligned}$$

Hence for all n, we have $y_n \leq r^{\,n-1} y_1$. Therefore

$$\sum_{n=1}^{\infty} x_n = \sum_{n=0}^{\infty} y_n \leq y_1 \sum_{n=0}^{\infty} r^{n-1} < \infty.$$

(b) Let $a > 1/2$. Then $\frac{1}{2a} < 1$, so there exists c such that $\frac{1}{2a} < c < 1$. Note that for $r = a(1+c)$,

$$r > a\left(1 + \frac{1}{2a}\right) = a + 1/2 > 1.$$

In this part, by similar reasoning, we assume

$$x_{n+1} > c\,x_n \text{ and } x_{2n} > a\,x_n \text{ for all } n.$$

Then for y_n as above,

$$\begin{aligned} y_{n+1} &= (x_{2^{n+1}} + x_{2^{n+1}+1}) + (x_{2^{n+1}+2} + x_{2^{n+1}+3}) + \cdots + (x_{2^{n+2}-2} + x_{2^{n+2}-1}) \\ &> (1+c)(x_{2^{n+1}} + x_{2^{n+1}+2} + x_{2^{n+1}+4} + \cdots + x_{2^{n+2}-2}) \\ &\geq a(1+c)(x_{2^n} + x_{2^n+1} + x_{2^n+2} + \cdots + x_{2^{n+1}-1}) \\ &= r\,y_n. \end{aligned}$$

Hence for all n, we have $y_n \geq r^{\,n-1} y_1$. Therefore

$$\sum_{n=1}^{\infty} x_n = \sum_{n=0}^{\infty} y_n \geq y_1 \sum_{n=0}^{\infty} r^{n-1} = \infty.$$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

If the sequence $(x_n)_{n\geq 1}$ is positive and monotone decreasing, then the problem follows from the Cauchy condensation test, which states that in that case, the series $\sum_{n=1}^{\infty} x_n$ and the series $\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} 2^k x_{2^k}$ converge or diverge simultaneously.

Thus, if $b < \frac{1}{2}$,

$$\frac{y_{n+1}}{y_n} = \frac{2^{n+1} x_{2^{n+1}}}{2^n x_{2^n}} = 2 \cdot \frac{x_{2\cdot 2^n}}{x_{2^n}} \leq 2 \cdot b < 2 \cdot \frac{1}{2} = 1,$$

so that by the ratio test, $\sum_{n=1}^{\infty} y_n$, and therefore, $\sum_{n=1}^{\infty} x_n$, converges. Analogously, if $a > \frac{1}{2}$, both series are divergent.

The condition $\frac{x_{n+1}}{x_n} \to 1$ as $n \to \infty$ implies that sequence $(x_n)_{n\geq 1}$ is weakly monotone, and by the reference below, this is sufficient to apply the Cauchy condensation test.

Reference: Liflyand, E., Tikhonov, S., Zeltser, M. (2011). Extending tests for convergence of number series. *J. Math. Anal. Appl.* 377: 194–206.

Also solved by PAUL BRACKEN, U. of Texas, Edinburg; ROBERT DOUCETTE, McNeese St. U.; JAMES DUEMMEL, Bellingham, WA; DMITRY FLEISCHMAN, Santa Monica, CA; RUSS GORDON, Whitman C.; EUGENE HERMAN, Grinnell C.; THE IOWA STATE UNDERGRADUATE PROBLEM SOLVING GROUP; ELIAS LAMPAKIS, Kiparissia, Greece; GUILLERMO ALVAREZ PARDO and ROBERT SCHWENNICKE, Cuesta C.; JOEL SCHLOSBERG, Bayside, NY; OMAR SONEBI; HONG BIAO ZENG, FORT HAYS ST. U.; AND THE PROPOSER.

If the sum of terms divided by the partial sums converges, then so does the sum of terms.

1157. *Proposed by So Mi Lim (student) and Sung Soo Kim, Hanyang University, Ansan, South Korea.*

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and let S_n denote the nth partial sum of this series. Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$ is convergent, then also $\sum_{n=1}^{\infty} a_n$ is convergent.

Solution by Jim Hartman, The College of Wooster, Wooster, OH.

We prove the contrapositive that if $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$ is divergent.

We note that both S_n and $T_n = \sum_{k=1}^{n} \frac{a_k}{S_k}$ are increasing, positive term sequences. Since $\lim_{n\to\infty} S_n = \infty$ there is a subsequence $\{n_k\}_{k=1}^{\infty}$ of the natural numbers so that $S_{n_{k+1}} > 2S_{n_k}$. We then have

$$
\begin{aligned}
T_{n_{k+1}} &= T_{n_k} + \sum_{p=n_k+1}^{n_{k+1}} \frac{a_p}{S_p} > T_{n_k} + \sum_{p=n_k+1}^{n_{k+1}} \frac{a_p}{S_{n_{k+1}}} \\
&= T_{n_k} + \frac{S_{n_{k+1}} - S_{n_k}}{S_{n_{k+1}}} > T_{n_k} + \frac{1}{2}
\end{aligned}
$$

since $\frac{S_{n_k}}{S_{n_{k+1}}} < \frac{1}{2}$. Since T_n is increasing, we must have $\lim_{n\to\infty} T_n = \infty$.

Editor's note: Budney, Han, and Vowe all pointed out that this problem is not new, providing, respectively, the following references.

1. B. M. Makarov, et al., *Selected Problems in Real Analysis*, AMS Translations of Mathematical Monographs, Vol. 107, 1992, Chapter 4, problem 2.11(a).
2. math.stackexchange.com/questions/388898
3. J. A. Ostrowski, Aufgabensammlung zur Infinitesimalrechnung, Basel, 1972, Vol. IIA, p. 56, problem 99, with solution in Vol. IIB, pp. 342–343.

Also solved by Michel Bataille, Rouen, France; Trevor Birenbaum and Sanjoy Kundu, U. of Miami; Paul Bracken, U. of Texas, Edinburg; Paul Budney, Sunderland, MA; Hongwei Chen, Christopher Newport U.; Sihan Chen, (student), Whitman C.; Richard Daquila, Muskingum U.; Robert Doucette, McNeese St. U.; James Duemmel, Bellingham, WA; Julia Gratton, Whitman C.; Lixing Han, U. of Michigan – Flint; Eugene Herman, Grinnell C.; The Iowa State Undergraduate Problem Solving Group; Walter Janous, Innsbruck, Austria; Ted Juste and C. J. Lungstrum; John Kieffer, U. of Minnesota Twin Cities; Elias Lampakis, Kiparissia, Greece; Kee-Wai Lau, Hong Kong, China; Northwestern U. Math Problem Solving Group; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Joel Schlosberg, Bayside, NY; Omar Sonebi; Sean Stewart, Bomaderry, NSW, Australia (2 solutions); Santiago Alzate Suárez (student), Universidad de Antioquia, Colombia; Michael Vowe, Therwil, Switzerland; Hong Biao Zeng, Fort Hays St. U.; and the proposer.

A constrained three-variable inequality

1158. *Proposed by Digby Smith, Mount Royal University, Alberta, Canada*

Suppose that a, b, and c are positive real numbers such that $ab + bc + ca = 3$. Prove that

$$(a^8+1)(b^8+1)(c^8+1)(a^4+1)(b^4+1)(c^4+1) \geq (a^2+1)^2(b^2+1)^2(c^2+1)^2.$$

When does equality hold?

Solution by Subhankar Gayen, West Bengal, India.

Using Hölder's inequality, we have

$$8\left(a^8+1\right) = \left(1^4+1^4\right)\left(1^4+1^4\right)\left(1^4+1^4\right)\left[\left(a^2\right)^4+1^4\right] \geq \left(a^2+1\right)^4,$$

and the Cauchy–Schwartz inequality gives

$$2\left(a^4+1\right) = \left(1^2+1^2\right)\left[\left(a^2\right)^2+1^2\right] \geq \left(a^2+1\right)^2.$$

Multiplying with similar inequalities gives

$$\sum_{\text{cyc}}\left(a^8+1\right)\left(a^4+1\right) \geq \frac{1}{2^{12}}\sum_{\text{cyc}}\left(a^2+1\right)^6.$$

Thus if suffices to show that

$$\left(a^2+1\right)\left(b^2+1\right)\left(c^2+1\right) \geq 8.$$

Again by the Cauchy–Schwarz inequality,

$$\begin{aligned} 2\left(a^2+1\right)\left(b^2+1\right)\left(c^2+1\right) &= \left(1^2+a^2+a^2+1^2\right)\left(1^2+b^2+c^2+b^2c^2\right) \\ &\geq (1+ab+ac+bc)^2 = 16, \end{aligned}$$

proving the inequality. Equality holds when $a = b = c = 1$.

Also solved by Michel Bataille, Rouen, France; Trevor Birenbaum, U. of Miami; Habib Far, Lone Star C. – Montgomery; Eugene Herman, Grinnell C.; Walter Janous, Innsbruck, Austria; Harris Kwong, SUNY Fredonia; Elias Lampakis, Kiparissia, Greece; Ioannis Sfikas, Athens, Greece; and the proposer.

An arctan inequality

1159. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*
Prove that for any real number $x > 0$, $\arctan(x) \cdot \arctan(\frac{1}{x}) > \frac{x}{2(x^2+1)}$.

Unfortunately, as several readers pointed out, this problem already appeared in the September 2019 issue of *Crux Mathematicorum*. We remind problem proposers to ensure that problems submitted to *The College Mathematics Journal* have not appeared in, and are not under consideration by another journal.

Solved by ADA U. PROBLEM SOLVING GROUP; MICHEL BATAILLE, Rouen, France; BRIAN BRADIE, Christopher Newport U.; HONGWEI CHEN, Christopher Newport U.; JOHN CHRISTOPHER, California St. U., Sacramento; BRUCE DAVIS, St. Louis Comm. C. at Florissant Valley; JAMES DUEMMEL, Bellingham, WA; BILL DUNN, Montgomery C.; CALEB EKE, Missouri St. U.; HABIB FAR, Lone Star C. – Montgomery; DMITRY FLESICHMAN, Santa Monica, CA; RAYMOND GREENWELL, Hofstra U.; J. A. GRZESIK, Allwave Corp.; LIXING HAN, U. of Michigan – Flint; EUGENE HERMAN, Grinnell C.; TOM JAGER, Calvin U.; WALTER JANOUS, Innsbruck, Austria; ELIAS LAMPAKIS, Kiparissia, Greece; MISSOURI ST. U. PROBLEM SOLVING GROUP; JULIO CESAR MOHNSAM, IFSul Campus Pelotas-RS, Brazil; NORTHWESTERN U. MATH PROBLEM SOLVING GROUP; IOANNIS SFIKAS, Athens, Greece; DIGBY SMITH, Mount Royal U., Calgary, AB, Canada; SEAN STEWART, Bomaderry, NSW, Australia; NORA THORNBER, Raritan Valley Comm. C.; MICHAEL VOWE, Therwil, Switzerland; STAN WAGON, Macalester C.; EDWARD WHITE and ROBERTA WHITE, Frostburg, MD; and the proposer.

A limit of a double sum of binomial coefficients

1160. *Proposed by Ángel Plaza and Pedro Jesús Rodríguez de Rivera (student), Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain.*
Evaluate the following limit:

$$\lim_{m\to\infty} \sum_{n\geq 0} \frac{1}{m^n} \sum_{k\geq 0} \frac{(-1)^k \binom{2n}{k}\binom{2k}{k}}{2^k}.$$

Solution by Robert Doucette, McNeese State University, Lake Charles, LA.
The coefficient of x^0 in $[(x^2+x^{-2})/2]^{2n}$ is $\binom{2n}{n}/2^{2n}$. Since

$$\begin{aligned}
\left(\frac{x^2+x^{-2}}{2}\right)^{2n} &= \left(1 - \frac{(x^1+x^{-1})^2}{2}\right)^{2n} \\
&= \sum_{k=0}^{2n} \binom{2n}{k} \left(-\frac{(x^1+x^{-1})^2}{2}\right)^k \\
&= \sum_{k=0}^{2n} \binom{2n}{k} \left(-\frac{1}{2}\right)^k (x^1+x^{-1})^{2k} \\
&= \sum_{k=0}^{2n} \binom{2n}{k} \left(-\frac{1}{2}\right)^k \sum_{i=0}^{2k} \binom{2k}{i} x^{2(k-i)} \\
&= \sum_{k=0}^{2n} \left(\sum_{i=0}^{2k} \binom{2n}{k}\binom{2k}{i} \left(-\frac{1}{2}\right)^k x^{2(k-i)}\right),
\end{aligned}$$

it follows that

$$\sum_{k=0}^{2n} \binom{2n}{k}\binom{2k}{k}\left(-\frac{1}{2}\right)^k = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}.$$

Using binomial series for $|x| < 1$,

$$\begin{aligned} 1/\sqrt{1-x} &= \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \binom{2n}{n} x^n \\ &= \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{x}{4}\right)^n. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n\geq 0} \frac{1}{m^n} \sum_{k\geq 0} \frac{(-1)^k \binom{2n}{k}\binom{2k}{k}}{2^k} &= \sum_{n\geq 0} \frac{1}{m^n} \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \\ &= \sum_{n\geq 0} \binom{2n}{n} \left(\frac{1}{4m}\right)^n = \frac{1}{\sqrt{1-\frac{1}{m}}}, \end{aligned}$$

and so the desired limit is 1.

Reference: Greene, D. H., Knuth, D. E. (1981). *Mathematics for the Analysis of Algorithms*, 3rd ed. (1990). Boston: Birkhäuser, Section 1.2.

Also solved by Michel Bataille, Rouen, France; Paul Bracken, U. of Texas, Edinburg; Brian Bradie, Christopher Newport U.; Dmitry Flesichman, Santa Monica, CA; Eugene Herman, Grinnell C.; Adebola Omotajo, Lone Star C. – Montgomery; Ioannis Sfikas, Athens, Greece; Michael Vowe, Therwil, Switzerland; and the proposer.

Editor's note: Previous issues contained the following errors.

- In the featured solution of problem 1092, the name of Panagiotis Krasopoulos was misspelled.
- The name of Christopher Jackson, Coleman, FL, was omitted from the list of solvers of problem 1151.
- The name of Herb Bailey (age 94), Rose Hulman Inst. Tech., was omitted from the list of solvers of problem 1153.

We apologize for the errors.

