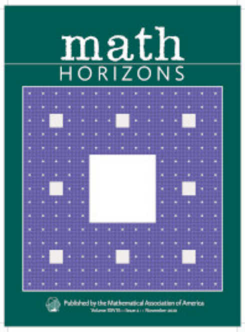

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The Playground

Welcome to the Playground! Playground rules are posted on page 33, except for the most important one: *Have fun!*

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THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Hexsquare (P410). Pierre Abbat (*bezitopo.org*) posed this three-part problem, based on familiar ways of stacking pennies. On the one hand, you can arrange 36 unit disks in a square pattern, but it takes one more disk to create a hexagonal pattern (see figure 1). On the other hand, 169 disks can be arranged as either a square or a hexagon. Call the distance between the horizontal lines indicated in figure 1 the *height* of the respective arrangements.

1) What is the difference in height between the square and hexagon arrangements of 169 disks?

2) Show there are infinitely many distinct numbers of disks that can be arranged in both ways.

3) What happens to the height difference between the two arrangements for the same number of disks, as that number goes to infinity?

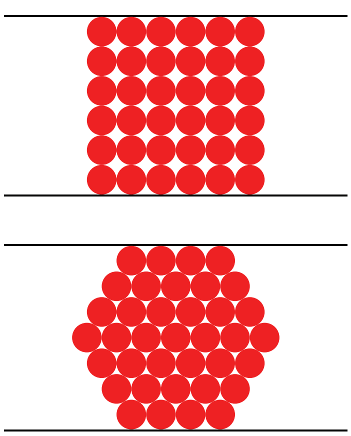


Figure 1. 36 disks arranged as a square and 37 arranged as a hexagon.

THE MONKEY BARS

These open-ended problems don't have a previously known exact solution, so we intend for readers to fool around with them. The Playground will publish the best submissions received (proofs encouraged but not required).

Dominoptimal (P411). Stan Wagon (Macalester College) extended last February's problem 398 ("101 Domino-tions") in yet a different direction (see P403 below for an earlier extension). It's not difficult to determine the least-perimeter rectangle into which you can pack n dominoes (1×2 rectangles) so that each domino has integer vertex coordinates and no two dominoes overlap. Problem 398 investigated packings with an additional "long-side restriction": no two dominoes are allowed to share an entire edge of length two. In that problem the minimal 14×15 rectangle has the same perimeter as it would without the additional constraint. However, in the case of $n = 2$, the dominoes fit into a 2×2 rectangle without the constraint but require a perimeter-10 rectangle with the long-side restriction (either 1×4 or 2×3, see figure 2). For which other n is a larger-perimeter rectangle required with the long-side restriction?

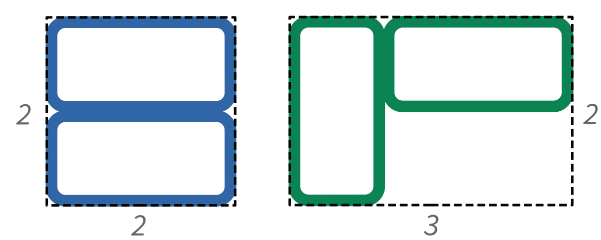


Figure 2. Packing two dominoes with (right) and without (left) the long-side restriction.

THE ZIP-LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip-Line problems require you to read the corresponding article, but doing so can never hurt, of course.

Flip Factors (P412). Cornelia Van Cott craftily left this problem lurking in her article "The Integer Hokey Pokey" (p. 24): A whole number A is called an *n-flip in base-b* (for $n > 1$) if the base-b representation of A is $(d_k d_{k-1} \ldots d_1 d_0)_b$, and of nA is $(d_0 d_1 \ldots d_{k-1} d_k)_b$. Prove that there are no prime n-flips in any base.

THE JUNGLE-GYM

Any type of problem may appear in the Jungle Gym—climb on!

Turn Dial (P413). Editor Emeritus Gary Gordon (Lafayette College) is back again, this time in collaboration with student Deniz Ozbay. A dial (as in figure 3) can point to any whole number from 1 to n, listed counterclockwise in order. The dial starts on 1, and then is turned in succession (always counterclockwise) one position, then two positions, then three positions, and so on until a final turn by $n - 1$ positions. For example, the first five positions for $n = 12$ will be 1, 2, 4, 7, 11, and 4. For what n will the dial point to a different position at every stage of this process?

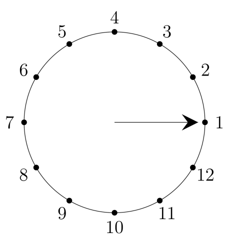


Figure 3. A dial with 12 positions.

APRIL WRAP-UP

Construction Challenge (P402). (From Friedrich Krauch, Zurich, Switzerland.) Two concentric circles c and d (with half the radius of c), and two distinct diameters of c, are given (see figure 4). An intersection point of two known (given or constructed) lines or circles is also considered known. A "step" consists of drawing a perpendicular to a known line through a known point, or a circle centered at a known point through another known point. In five steps, construct one side of a regular pentagon inscribed in circle c.

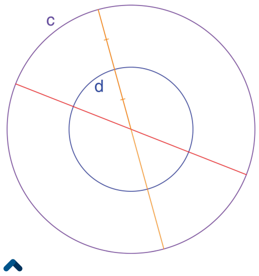


Figure 4. Soon to be a pentagon?

The Playground received a solution from Randy K. Schwartz (Schoolcraft College) and the beautifully illustrated one from the CNU Math Circle shown in figure 5. The steps are:

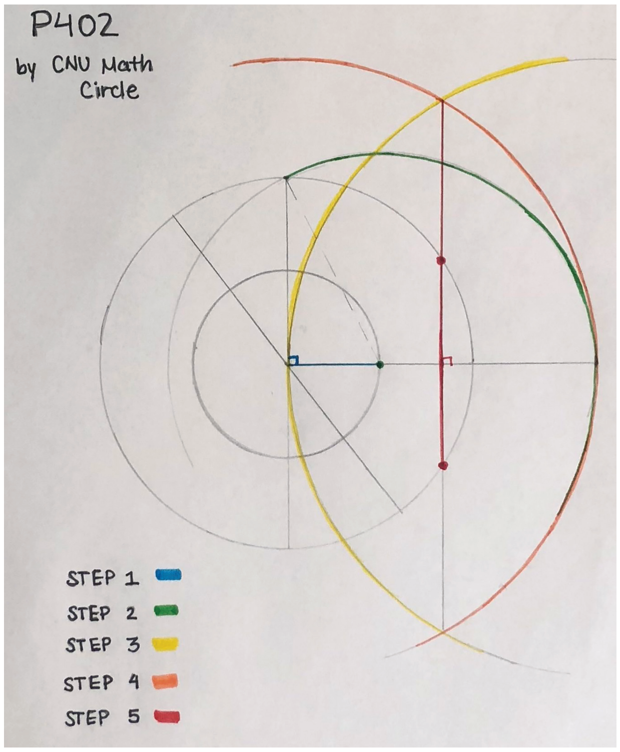


Figure 5. Completing the pentagon.

THE CAROUSEL—OLDIES, BUT GOODIES

In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

P.S.P.I. (C31). For a change of pace, Seljon Akhmedli (North Dakota State University) proposed an original twist on a previous Carousel problem (C18 "Squared Triangle Cubes"). Call a set $\{(j, l), (k, m)\}$ of two ordered pairs of whole numbers a PSPI (for Power of Sums of Powers Identity) if for all n,

$$(1^j + 2^j + \cdots + n^j)^l = (1^k + 2^k + \cdots + n^k)^m.$$

Note that if $\{(j, l), (k, m)\}$ is a PSPI, then any "power" $\{(j, al), (k, am)\}$ of it is also a PSPI. Are there any PSPIs other than powers of $\{(1,2), (3,1)\}$, the one identified in Problem C18?

1) Take the perpendicular p to the first diameter through the intersection O of the two diameters. Choose a point P where p intersects circle d (blue).
2) Draw the circle centered at P through the intersection of the first diameter and circle c (green). Let Q be the point where this circle intersects p on the same side of the first diameter as P.
3) Draw the circle centered at Q through O (yellow).
4) Draw the circle centered at O through Q (orange).
5) Take the perpendicular q to p through either/both of the intersection points of the latter two circles (red).

The two points of intersection of q with c define the desired side of an inscribed pentagon. Indeed, letting the intersection of p and q be R, compute $\overline{OR} = \phi,$ the golden ratio, whence the central angle subtended by these two points is $\tau / 5\,(= 2\pi / 5)$ as desired.

Randy shaves one step off this solution by noting that the point S where p intersects c is equidistant from Q and the two desired endpoints of the side, so a single circle centered at S will suffice instead of the last two. The problem author notes there is also a five-step construction using only auxiliary points in the interior of the original circle c. We will be happy to publish any reader-submitted solution meeting this further criterion as an addendum.

Dense Dominoes (P403). A *double-n domino* is a 1×2 rectangle with each of its two 1×1 squares labeled by a nonnegative integer no larger than n. There are $(n+1)(n+2)/2$ distinct double-n dominoes (e.g. 28 double-6 dominoes). A *proper packing* of a set of dominoes is a nonoverlapping assignment of each domino to a position with integer vertex coordinates, such that whenever two dominoes share an edge segment, their adjacent squares have the same label.

What is the least-perimeter rectangle that can contain a proper packing of all double-n dominoes?

We received analysis and conjectures from the same dynamic domino duo, Evan Ganning and Meagan Praul (Seton Hall University), who provided the featured solution for P398.

Consider the area A of a rectangle containing all double-n dominoes. Clearly, this area must exceed the area of the dominoes themselves, so $A > (n+1)(n+2)$. But, as no two dominoes are alike, none can share a long edge. Hence the two squares adjacent to either long edge of the $n(n+1)/2$ dominoes with two different numbers (the "non-doubles") cannot both be filled (as the two dominoes filling those squares would not obey the matching rule). Thus, there must be an empty square—what Meagan and Evan call a "gap" when it occurs within the rectangle—adjacent to each such long side.

On the other hand, if the perimeter of the layout is P, then up to $P/2$ of the dominoes could have a long edge on the perimeter, avoiding one gap. Also, any empty interior square could serve as the gap for up to four dominoes. Putting these facts together, we see

$$\begin{aligned} A &\geq (n+1)(n+2) + \frac{n(n+1) - P/2}{4} \\ &> (n+1)(n+2) + n(n+1)/8 \\ &= (n+1)(9n+16)/8. \end{aligned}$$

Because a square has the least perimeter for a given area, we have that $P \geq 2\left\lceil 2\sqrt{A}\right\rceil$. Combining these inequalities provides the best proven lower bound for P in terms of n.

However, in practice the best arrangements that Evan and Meagan could find have at least one gap for each of the non-double dominoes (and intriguingly in many cases, exactly one gap for each). Following the above outline with precisely one gap per non-double domino leads to their conjecture that the optimal perimeter is

$$P = 2\left\lceil 2\sqrt{\frac{(n+1)(3n+4)}{2}}\right\rceil.$$

The team submitted arrangements for $n = 3$ through $n = 9$ achieving this value, a few of which are shown in figure 6. None of their arrangements were proven to be optimal.

Figure 6. Packings of double-n dominoes for $n = 3$, 4, 7, and 9. Note the middle two cases have exactly one gap per non-double domino.

The Playground would gladly publish future submissions of any arrangements proved optimal or with a smaller perimeter than this conjectured minimum. Concerning the latter possibility, note that the conjectured minimum area for $n = 5$ is 57 (yielding $P = 32$). Eliminating just one of 15 gaps would, on the contrary, allow packing into a 7×8 rectangle.

Everything Affine (P404). This problem took its inspiration from Hsu, Ostroff, and Van Meter's article "Projectivizing SET." The *n-dimensional affine space* over $\mathbb{F}_3$ consists of all n-tuples of integers modulo 3. A *line* in this geometry is a set of three such tuples that sum to 0 (modulo 3) in every component. Exactly how many lines are there in the n-dimensional affine space over $\mathbb{F}_3$?

We received a generalized solution (for any prime in place of 3) from the Missouri State University problem solvers, a solution from Randy Schwartz, and a partial solution from Dmitry Fleischman.

Note that there are 3^n points in the affine space. If we choose any two points, there is exactly one additional point that completes a line, by the sum condition on the components. On the other hand, for any line, there are three possible pairs of points we might choose leading to that line (we could leave any one of the three points out). Thus, the number of lines is

$$\binom{3^n}{2} / \binom{3}{2} = 3^{n-1}(3^n - 1) / 2.$$

Fairly Unfair (P405). (Zhi Chao Li, UT Austin.) It is relatively well known that there is no procedure by which a fair coin can be tossed a finite number of times to fairly choose with equal likelihood among seven different alternatives. Determine, with proof, whether there is an *unfair* coin that may be tossed a finite number of times to choose among seven alternatives with equal likelihood.

The Playground received no submissions for this problem; it will be held open through the submission deadline for the current problems.

CAROUSEL SOLUTION

Suppose $\{(j, l), (k, m)\}$ is a PSPI. Without loss of generality we can suppose $j < k$, whence $l > m$, and, by taking roots, that l and m have no common factors. Then taking $n = 2$, $(1+2^j)^l = (1+2^k)^m$. Modulo 2^{j+1}, this is $1 + l \cdot 2^j \equiv 1$, so l is even. Therefore, m is odd, which means $1 + 2^k$ is a perfect square.

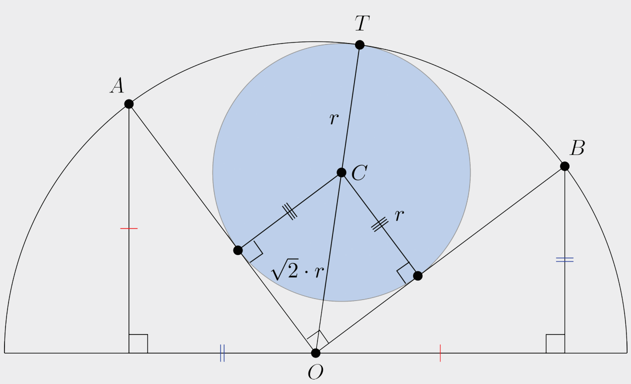
In particular, $2^k = r^2 - 1 = (r-1)(r+1)$, and thus both $r-1$ and $r+1$ are powers of two. The only possibility is $r = 3$, which corresponds to $k = 3$. Thus, $(1+2^j)^{l/2} = 3^m$ with m odd, which yields $1 + 2^j \equiv 3 \pmod 4$, forcing $j = 1$. Therefore, we conclude that every PSPI is a power of the familiar $\{(1,2), (3,1)\}$.

SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems and solutions should be submitted to MHproblems@maa.org and MHsolutions@maa.org, respectively (PDF format preferred). Paper submissions can be sent to Glen Whitney, UCLA Math Dept., 520 Portola Plaza MS 6363, Los Angeles, CA 90095. Please include your name, email address, and school affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is January 15, 2021.

Solution to Geometry Puzzle from page 2.



If the radius of the circle is r, the radius of the semicircle is $\sqrt{2} \cdot r + r$. Thus, the circle covers $2 / (\sqrt{2}+1)^2$ of the semicircle.

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