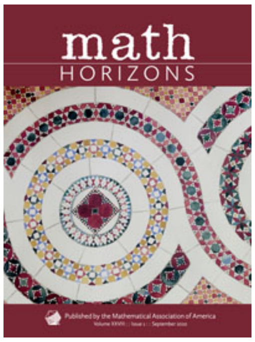

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The Playground

Welcome to the Playground! Playground rules are posted on page 33, except for the most important one: *Have fun!*

Glen Whitney

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THE PLAYGROUND

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THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Half a Quarter (P406). Here's another geometry gem from Arsalan Wares (Valdosta State University). Quarter circle $\overset{\frown}{AQC}$ has center B as in figure 1. Unit semicircle CPQ is tangent to BA at P. What is the radius of $\overset{\frown}{AQC}$?

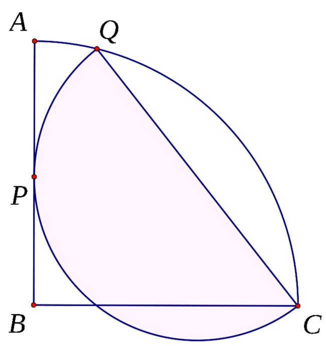

Figure 1. A semicircle almost inscribed in a quarter circle.

THE MONKEY BARS

These open-ended problems don't have a previously known exact solution, so we intend for readers to fool around with them. The Playground will publish the best submissions received (proofs encouraged but not required).

Buffon Encore (P407). The Comte de Buffon famously showed that the probability of a randomly dropped unit needle crossing one of the lines on a plane ruled with parallel lines one unit apart is $2/\pi$. On the one hand, problem 397 "Cramped Buffon" (Nov. 2019) revealed that this probability cannot be recreated by a single line down the center of a rectangular sheet: the probability a needle crosses this line, conditioned on the needle lying fully on the sheet, is always less than $2/\pi$, regardless of the sheet's width.

On the other hand, if a finite sheet is ruled with numerous parallel lines ϵ apart, the probability that a unit needle landing on the page crosses a line clearly exceeds $2/\pi$. What is the minimum number of lines that can be placed on a finite sheet to yield a probability of $2/\pi$ that a unit needle randomly dropped on the sheet crosses a line? The Playground will publish the best couple of configurations received, with ties broken by the smallest total width of the sheet.

THE ZIP-LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip-Line problems require you to read the corresponding article, but doing so can never hurt, of course.

Conic-structions (P408). The table of geometric solutions of cubic equations in Siadat's and Tholen's article "Omar Khayyam: Geometric Algebra and Cubic Equations" (page 13) uses only four of the six possible pairs of the three conics considered. It is missing intersections of two circles or of two parabolas.

Suppose you were restricted to only being able to draw conics in the plane of one type (either circles or parabolas) and find their points of intersection. Would you be able to solve any cubic equations? To make this concrete, for each type of conic (circle or parabola), either find two instances of that conic (with rational coefficients) that intersect at a point with x-coordinate equal to the cube root of 2, or show that no such pair exists.

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

More Leeway (P409). Prof. Bill Dunham of Bryn Mawr posed this question of angular possibilities: The large triangle in figure 2 is broken into seven component triangles. The degree measures of all of the internal angles of the configuration are specified, except for the four labeled as α, β, γ, and δ in the figure. What are all the possible values for the angle sum $\alpha + \beta + \gamma + \delta$?

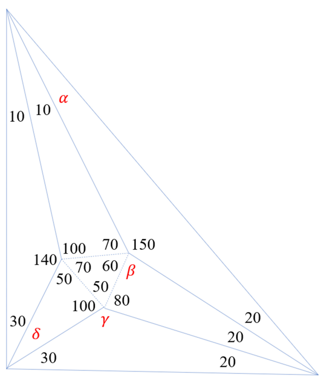

Figure 2. A triangle partitioned into seven smaller triangles with four unknown angles.

THE CAROUSEL—OLDIES, BUT GOODIES

In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

Zip Pick (C30). Prof. Joseph S. is unsure where to go on vacation. So, he writes all of the possible ZIP codes in order from 00000 to 99999 in a large circle. He starts at 00001, crosses it out, skips one, crosses the next one out, skips one, and so on, always skipping the next uncrossed ZIP code around the circle and crossing the next uncrossed one out. He decides to visit the last ZIP code that remains not crossed out in this process. Where does Prof. Joseph S. end up going for vacation?

FEBRUARY WRAP-UP

Pell Tangent (P393). This problem from Ángel Plaza De La Hoz originally appeared with a typo, noted by Alexander Karabegov (Abilene Christian University), in which the "+1" in the first denominator of the series was replaced by "–1."

Let P_k be the kth Pell number, i.e., the denominator produced by cutting off the infinite repeating continued fraction $\sqrt{2} = [1; \overline{2}]$ after k terms. By convention, $P_0 = 0$ and $P_1 = 1$. Compute

$$\sum_{k=1}^{\infty} \arctan \frac{P_k + P_{k-1}}{P_k P_{k+1} + 1} \arctan \frac{P_{k+1} + P_k}{P_k P_{k+1} - 1}.$$

Solutions from Alexander and Randy K. Schwartz (Schoolcraft College) demonstrate that the value of this series, which we will denote S, is $\boldsymbol{\pi^2 / 16}$.

First, obtain the usual recurrence for the Pell numbers, namely $P_{k+1} = 2P_k + P_{k-1}$, and use it to rewrite the summand of S as

$$\arctan \frac{P_{k+1} - P_k}{P_k P_{k+1} + 1} \arctan \frac{P_{k+1} + P_k}{P_k P_{k+1} - 1}.$$

Now, choose a_k so that $\tan a_k = 1 / P_k$. Then

$$\frac{P_{k+1} - P_k}{P_k P_{k+1} + 1} = \frac{\tan a_k - \tan a_{k+1}}{1 + \tan a_k \tan a_{k+1}} = \tan(a_k - a_{k+1}),$$

and similarly for the other fraction in the summand with – and + reversed. Check that the a_k are all in the first quadrant to verify that the arctangents in the summand cancel the tangents. Finally, note that the a_k approach 0 to see that

$$S = \sum_{k=1}^{\infty} a_k^2 - a_{k+1}^2 = a_1^2.$$

But $P_1 = 1$, so $a_1 = \pi / 4$ and $S = \pi^2 / 16$.

Randy also noted that there is strong numerical evidence that the series with the typo also converges, surprisingly to exactly $2S$; the Playground would gladly publish a proof of this other sum if anyone is able to provide one.

101 Domino-tions? (P398). For the purposes of this problem, a domino is simply a 1×2 rectangle that must be placed in the plane so that all of its vertices have integer coordinates. What is the smallest rectangle, by perimeter, into which 101 nonoverlapping dominoes may be placed so that no two dominoes share an entire long edge?

Photograph by Meagan Praul.

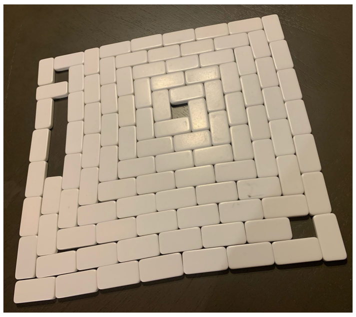

Figure 3. 101 dominos arranged in a rectangle.

We received solutions from Gabriel Augusto Correia (Universidade de Brasília), Evan Ganning and Meagan Praul (Seton Hall University), Tucker Germain, Kabrie Karschner, and Micah Wheeler (Taylor University), Vasile Teodorovici (NSERC Canada), Stan Wagon (Macalaster College), and the Armstrong, Cal Poly Pomona, Missouri State University, and Skidmore College problem-solving groups. We also received a partial solution from Josh Harden, Lauren James, and Autumn Thompson of Taylor University.

The minimum possible perimeter is 58. To see this, note first that the rectangle must have sufficient area to fit all 101 dominos, or at least 202 square units. The minimum perimeter rectangle for a given area is a square, but a 14×14 square (with perimeter 56) is slightly too small. Thus, the smallest possible perimeter is 58, and indeed, as the photo in figure 3 from Meagan and Evan shows, it is possible to pack all 101 dominos into a 14×15 rectangle with this perimeter. The groups from Armstrong State, Missouri State, and Seton Hall also noted it is possible to pack them into a 13×16 rectangle, and that a 12×17 rectangle apparently has barely enough area, but is nevertheless unworkable because of the long-side non-adjacency rule.

Tri-Hex (P399). From Arsalan Wares (Valdosta State University): Regular hexagon A and equilateral triangle B, both of unit area, overlap as shown in figure 4. What is the (shaded) area of overlap?

In addition to the most direct solution from David Morgan and Maria Stuebner of the CNU Math Circle presented here, we received a variety of other successful attacks on this problem from Brian Beasley (Presbyterian College), Cathryn Dunn, Sara Horn, and Jeff Jewett (Taylor University), Dmitry Fleischman, Randy Schwartz, Farid Taghiyev (ADA University, Azerbaijan), Vasile Teodorovici, and the Cal Poly Pomona problem solving group.

The area of overlap is $\mathbf{(5-\sqrt{3})/4}$. We decompose this area into the equilateral triangle $A_1A_3A_5$ of area 1/2 and three copies of triangle $\mathcal{I} = A_1A_5C_3$ (see figure 4). The first step in computing area $\mathcal{I}$ is to determine $\angle A_1A_5C_3$. Let D be the reflection of A_6 in A_1A_5; then D is also the center of hexagon A. Because $A_6A_1DA_5$ is a rhombus, A_6D is perpendicular to A_1A_5. And since

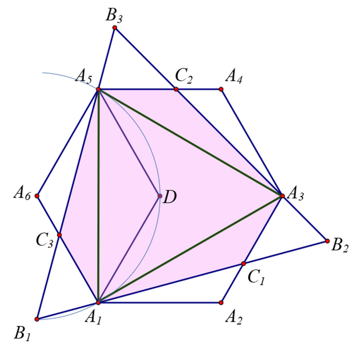

Figure 4. An overlapping hexagon and triangle of equal area.

both $A_6A_1A_5$ and $B_1A_1A_5$ have area 1/6, segment A_6B_1 is parallel to A_1A_5.

Hence, B_1A_6D is a right angle. Moreover, quadrilateral $B_1A_1DA_5$ is cyclic (with center A_6), as it has supplementary opposite angles at B_1 and D. Thus, arc B_1A_1 measures $\tau/4-\tau/6=\tau/12$ (where $\tau = 2\pi$ is the measure of a full circle). So $\angle A_1A_5C_3 = \tau/24$, and is the bisector of $\angle A_1A_5A_6$. Now, let area $\mathcal{A} = A_6C_3A_5$. By the bisector theorem,

$$\mathcal{A}/\mathcal{I} = \overline{A_6C_3}/\overline{C_3A_1} = \overline{A_6A_5}/\overline{A_1A_5} = 1/\sqrt{3}.$$

Together with $\mathcal{A}+\mathcal{I} = 1/6$, this yields $\mathcal{I} = 1/(6+2\sqrt{3})$, and we calculate the overlap $1/2+3\mathcal{I} = (5-\sqrt{3})/4$.

Pyramix (P400). In his article “Tricolor Pyramids,” Jacob Siehler investigated three-colored triangular arrays in which every triad composed of a cell and the two below it is either monochrome or contains all three colors. Suppose a size n pyramid is colored in a non-monochrome fashion according to this rule. Find, with proof, the maximum number of any one color of cells that the pyramid can contain.

We received the solution presented from the Missouri State Problem Solving Group, and partial solutions from Randy Schwartz and from the Skidmore College problem solvers. On the one hand, a non-monochrome triangle can have at most $n(n-1)/2$ cells of one color. Note that by making the top $n-1$ rows monochrome and alternating the other two colors on the bottom row, we can achieve a non-monochrome triangle of size n with $n(n-1)/2$ cells of one color.

On the other hand, if there are $1+n(n-1)/2$ cells of color c, then there is at least one row entirely colored c (as otherwise at most $0+1+2+\cdots+(n-2)+(n-1)=n(n-1)/2$ cells can be color c). So, let k be the longest row that is monochrome colored c. All of the shorter rows must be color c, by the coloring rule, and none of row $(k+1)$ can be color c (or else they would all be c). If $k < n$, that would leave

$$1+(k+1)+(k+2)+\cdots+(n-2)+(n-1)$$

cells of color c in $(k+2)+(k+3)+\cdots+(n-1)+n$ positions, so again we would have to have a (longer) row entirely colored c, contradicting the choice of k. We conclude $k = n$, i.e., the triangle is monochrome.

Equitable Marbles (P401). This problem was submitted by Matt Enlow (Dana Hall School, Wellesley, MA).

I have a bag of 100 marbles of three different colors. If you were to reach in and grab three of the marbles at random, there's a 20% chance that you would pull out one of each color. How many marbles of each color are in my bag?

The Playground received a baker's dozen responses to this problem, including solutions from Michael Ask (Ulysses High School), Brian Beasley, Dmitry Fleischman, Tarlan Ismayilsoy (ADA University, Azerbaijan), Kayla Nicolich and Leslie Rodriguez (Seton Hall University), Randy Schwartz, Vasile Teodorovici, and the Armstrong, Cal Poly Pomona, Missouri State, and Skidmore College problem-solving groups, as well as partial solutions from two groups of collaborators at Taylor University: Sarah Gorski, Rebekah Griggs, and Flora Wang; and Jacob Hockett, Scott Mitchell, and Clay Vander Kolk.

There are **21, 35,** and **44** marbles of the three colors. Largely following the (similar) Armstrong and Missouri solutions, let a, b, and c represent the numbers of marbles of each color, so that $a + b + c = 100$. There are abc ways to choose one marble of each color, which must account for 1/5 of the

$$\binom{100}{3}=161{,}700$$

ways to choose any three marbles, meaning $abc = 32{,}340$.

For fixed a, the product bc is maximized when $b = c$, whence $a(100-a)^2 \geq 4\cdot 32{,}340$. By color symmetry, this bounds all of a, b, and c between 21 and 48, inclusive. Now note the prime factorization of $32{,}340 = 2^2\cdot 3\cdot 5\cdot 7^2\cdot 11$; each of these prime factors must be distributed into one of a, b, or c.

Because a, b, and c are less than 49, we immediately have that 7, 7, and 11 must be distributed into different summands. Then by considering where the factor of 5 goes, one of the summands must be exactly 35, leaving the other two with product 924 and sum 65. This latter pair has the unique solution {21, 44}, as desired.

CAROUSEL SOLUTION

Rather than take the time to write 100,000 ZIP codes in a circle, we recommend finding a recurrence for $g(n)$, the last number remaining when 0 to $n-1$ are written and crossed out in this fashion. If n is even, then in the first pass, all of the odd numbers are crossed out, leaving just half as many numbers (with double their "usual" values) to undergo the same process. In other words, $g(2k) = 2g(k)$. In the odd case, something similar happens except that the 0 is also crossed out, leaving the remaining numbers also 2 larger than "usual," so $g(2k+1) = 2g(k)+2$.

Computing the first several values of g leads to the guess $g(n)=2n-t(2n)$, where $t(m)$ is the largest power of 2 less than or equal to m. This formula can be verified by induction. Hence the last remaining ZIP code is $g(100{,}000) = 200{,}000 - 131{,}072 = 68928$, or Bladen, Nebraska—coincidentally less than 35 miles from the geographic center of the contiguous United States. Note that this "Josephus Problem," as it's called, becomes much more delicate when a fixed number *larger* than one of uncrossed numbers is skipped between every crossing out.

SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems and solutions should be submitted to MHproblems@maa.org and MHsolutions@maa.org, respectively (PDF format preferred). Paper submissions can be sent to Glen Whitney, UCLA Math Dept., 520 Portola Plaza MS 6363, Los Angeles, CA 90095. Please include your name, email address, and school affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is October 31, 2020.

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