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Problems and Solutions

PROBLEMS

Proposals

To be considered for publication, solutions should be received by November 1, 2020.

2096. *Proposed by H. A. ShahAli, Tehran, Iran.*

Any three distinct vertices of a polytope P form a triangle. How many of these triangles are isosceles if P is

(a) a regular n-gon?

(b) one of the Platonic solids?

(c) an n-dimensional cube?

2097. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

For a real number $x \notin \frac{1}{2} + \mathbb{Z}$, denote the nearest integer to x by $\langle x \rangle$. For any real number x, denote the largest integer smaller than or equal to x and the smallest integer larger than or equal to x by $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively. For a positive integer n let

$$a_n = \frac{2}{\langle \sqrt{n} \rangle} - \frac{1}{\lfloor \sqrt{n} \rfloor} - \frac{1}{\lceil \sqrt{n} \rceil}.$$

(a) Prove that the series $\sum_{n=1}^{\infty} a_n$ is convergent and find its sum L.

(b) Prove that the set

$$\left\{ \sqrt{n} \left(\sum_{k=1}^{n} a_k - L \right) : n \geq 1 \right\}$$

is dense in [0, 1].

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We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com. More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to mathmagproblems@maa.org.

2098. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Let $Z_0 = 0$, $Z_1 = 1$, and recursively define random variables $Z_2, Z_3, \ldots,$ taking values in $[0, 1]$ as follows: For each positive integer k, Z_{2k} is chosen uniformly in $[Z_{2k-2}, Z_{2k-1}]$ and Z_{2k+1} is chosen uniformly in $[Z_{2k}, Z_{2k-1}]$.

Prove that, with probability 1, the limit $Z^* = \lim_{n\to\infty} Z_n$ exists and find its distribution.

2099. *Proposed by Russ Gordon, Whitman College, Walla Walla, WA and George Stoica, Saint John, NB, Canada.*

Let r and s be distinct nonzero rational numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

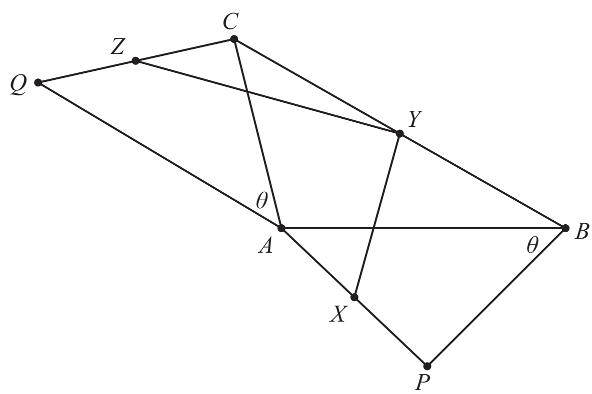
$$f\left(\frac{x+y}{r}\right) = \frac{f(x) + f(y)}{s}$$

for all real numbers x and y.

2100. *Proposed by Yevgenya Movshovich and John E. Wetzel, University of Illinois, Urbana, IL.*

Given $\triangle ABC$ and an angle θ, two congruent triangles $\triangle ABP$ and $\triangle QAC$ are constructed as follows: $AQ = AB$, $BP = AC$, $m\angle ABP = m\angle CAQ = \theta$, B and Q are on opposite sides of $\overleftrightarrow{AC}$, and C and P are on opposite sides of $\overleftrightarrow{AB}$, as shown in the figure. Let X, Y, and Z be the midpoints of segments AP, BC, and CQ, respectively.

Show that $\angle XYZ$ is a right angle.



Quickies

1101. *Proposed by Robert Calcaterra, University of Wisconsin, Platteville, WI.*

It is well known that it is impossible to square a circle using just a straightedge and compass. In other words, given a circle, it is impossible to construct a square having the same area as the given circle.

Given an arbitrary polygon, is it possible to construct a square with the same area using only straightedge and compass?

1102. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $n \geq 1$ be an integer. Calculate

$$I_n = \int_0^1 \frac{\ln(1-x) + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}}{x^{n+1}}\, dx.$$

Solutions

Units in a familiar ring with unfamiliar multiplication **June 2019**

2071. *Proposed by Ioan Băetu, Botoşani, Romania.*

Let $n > 1$ be an integer, and let $\mathbb{Z}_n$ be the ring of integers modulo n. For fixed $k \in \mathbb{Z}_n - \{0\}$, define a binary operation "$\circ$" on $\mathbb{Z}_n$ by $x \circ y = (x-k)(y-k) + k$ for all $x, y \in \mathbb{Z}_n$. Let U be the group of units of $\mathbb{Z}_n$ (under multiplication), and let U_k° be the set of elements of $\mathbb{Z}_n$ that are invertible under the operation $\circ$. Characterize those n with the property that $U \neq U_k^\circ$ for all $k \in \mathbb{Z}_n - \{0\}$.

Solution by Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.
We claim that n has the desired property if and only if it is square-free.

First note that $x \circ (1+k) = x$ for all $x \in \mathbb{Z}_n$, so $1+k$ is the identity for $\circ$. If $u \in U$, then

$$(k+u) \circ (k+u^{-1}) = uu^{-1} + k = 1 + k,$$

so

$$k + U = \{k + u \mid u \in U\} \subseteq U_k^\circ.$$

Conversely, if $v \in U_k^\circ$, then there is a $w \in \mathbb{Z}_n$ with

$$1 + k = v \circ w = (v-k)(w-k) + k$$

and thus $1 = (v-k)(w-k)$. So $v - k \in U$ and hence $U_k^\circ = k + U$.

Suppose n is not square-free and let k be the product of all the distinct prime factors of n, so $k \in \mathbb{Z}_n - \{0\}$. If a is any integer, then $\gcd(a, n) = 1$ if and only if $\gcd(k + a, n) = 1$. It follows that $U = k + U = U_k^\circ$ and hence n does not have the desired property.

Suppose now that n is square-free. If $k \in \mathbb{Z}_n - \{0\}$ then at least one of the prime factors of n does not divide k; let q be the product of all the prime factors of n that do not divide k. Then $\gcd(q - k, n) = 1$ and hence $q - k \in U$. But then

$$q = k + (q-k) \in k + U = U_k^\circ,$$

and since $q \notin U$ then we have $U \neq U_k^\circ$ for all $k \in \mathbb{Z}_n - \{0\}$. So n has the desired property.

Also solved by Hafez Al-Assad (Syria), Anthony J. Bevelacqua, Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia), Robert Calcaterra, Ali Deeb (Syria), Briana Foster-Greenwood, Tom Jager, Peter McPolin (Northern Ireland), and the proposer. There was one incomplete or incorrect solution.

Two initial value problems **June 2019**

2072. *Proposed by Julien Sorel, Piatra Neamt, PNI, Romania.*

(a) Show that the initial value problem

$$\begin{cases} y' = \sqrt{1-y^2}, \\ y(0) = 1 \end{cases}$$

has infinitely many solutions defined on $\mathbb{R}$.

(b) By contrast, show that the initial value problem

$$\begin{cases} y' = \sqrt{x^2-y^2}, \\ y(1) = 1 \end{cases}$$

has no solutions defined on an open interval containing $x = 1$.

Solution by Ali Deeb (student) and Hafez Al-Assad (student), Higher Institute for Applied Sciences and Technology, Damascus, Syria.

(a) First note that any solution with $y(b) = 1$ must have $y(x) = 1$ for all $x \geq b$. This follows from the fact that y is nondecreasing, since $y' \geq 0$, and that $-1 \leq y \leq 1$. Similarly, if $y(a) = -1$, then $y(x) = -1$ for all $x \leq a$.

For any $k \geq \pi/2$, let

$$y(x) = \begin{cases} -1 & x < -k - \frac{\pi}{2} \\ \sin(x+k) & -k - \frac{\pi}{2} \leq x < -k + \frac{\pi}{2} \\ 1 & x \geq -k + \frac{\pi}{2} \end{cases}$$

It is straightforward to check that y is continuous, differentiable, and satisfies the given differential equation.

(b) Suppose to the contrary that such a $y(x)$ exists. For x close to 1, $y(x)$ is also close to 1. In particular, they are both positive. The condition $y' = \sqrt{x^2 - y^2}$ forces $x \geq y(x) > 0$. Consider the function

$$g(x) = \frac{1 - y(x)}{1 - x}$$

defined for $x < 1$ and x close to 1. We have

$$\lim_{x \to 1^-} g(x) = y'(1) = \sqrt{1^2 - y(1)^2} = 0.$$

In particular, $g(x) < 1$ for x sufficiently close to 1. But this gives

$$\frac{1 - y(x)}{1 - x} < 1 \Rightarrow 1 - y(x) < 1 - x \Rightarrow y(x) > x$$

resulting in a contradiction.

Also solved by Yagub Aliyev (Azerbaijan), Michel Bataille (France), Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), David M. Bradley, Robert Calcaterra, Bruce E. Davis, John N. Fitch, Lixing Han, Eugene A. Herman, Tom Jager, Kee-Wai Lau (Hong Kong), Albert Natian, José Nieto (Venezuela), Northwestern University Math Problem Solving Group, Sonebi Omar (Morocco), Sung Hee Park (South Korea), Francisco Perdomo & Ángel Plaza (Spain), Edward Schmeichel, Nicholas C. Singer, Lawrence R. Weill, Xinyi Zhang (Canada), and the proposer. There were two incomplete or incorrect solutions.

Unambiguous factorial expansions **June 2019**

2073. *Proposed by Enrique Treviño, Lake Forest College, Lake Forest, IL.*

A *factorial expansion* is any formal expression of the form

$$\overline{a_k a_{k-1} \dots a_2 a_1}$$

where $a_1, a_2, \dots, a_k$ are k integers ($k \geq 1$) such that $0 \leq a_i \leq i$ for $i = 1, 2, \dots, k$. The *value* of such a factorial expansion is

$$a_k \cdot k! + a_{k-1} \cdot (k-1)! + \cdots + a_2 \cdot 2! + a_1 \cdot 1!.$$

If the integers $a_1, \dots, a_k$ are expressed in base 10 and their digits simply written together without separation, the value of the factorial expansion so written is often ambiguous. For instance, the expansion $\overline{10000000000}$ may be interpreted as having coefficients 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 and value $1 \times 11! + 0 \times (10! + 9! + \ldots + 1!) = 11!$, or having coefficients 10, 0, 0, 0, 0, 0, 0, 0, 0, 0 and value $10 \times 10! + 0 \times (9! + 8! + 7! + \cdots + 1!) = 10 \times 10!$. Such factorial expansions are called *ambiguous.* On the other hand, some factorial expansions are unambiguous: for example, the expansion $\overline{311}$ must have the value $3 \times 3! + 1 \times 2! + 1 \times 1! = 21$. Prove that there are only finitely many unambiguous factorial expansions, and find the one whose value is largest.

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.
Let us call the *length* of a factorial expansion $\overline{a_k a_{k-1} \dots a_2 a_1}$ the total number of decimal digits in the a_i's. We claim that any factorial expansion with length greater than 99 is ambiguous. Indeed, let $d_n d_{n-1} \dots d_1$ be the sequence of digits in such an expansion. Then it may be interpreted as

$$\overline{d_n d_{n-1} \dots d_1} \text{ or as } \overline{(10d_n + d_{n-1}) d_{n-2} \dots d_1},$$

and the first has greater value than the second. Therefore there are only finitely many unambiguous factorial expansions.

We claim that the unambiguous factorial expansion whose value is largest is

$$m = \overline{\underbrace{99\ldots99}_{91 \text{ nines}}87654321},$$

whose value is

$$M = \sum_{k=1}^{8} k \cdot k! + 9 \sum_{k=9}^{99} k!.$$

Indeed, the value of a factorial expansion $\overline{a_k a_{k-1} \ldots a_2 a_1}$ is

$$\sum_{j=1}^{k} a_k \cdot k! \leq \sum_{j=1}^{k} k \cdot k! = (k+1)! - 1,$$

thus if $k < 99$ its value is less than 99! and less than M. If $k \geq 99$, to be unambiguous we must have $k = 99$ and all the a_i's must be digits. Then the greatest value is clearly attained with m.

Also solved by Hafez Al-Assad (Syria), Robert Calcaterra, Ali Deeb (Syria), Kelly Jahns, Vasile Teodorovici (Canada), and the proposer.

Zeta(2) in disguise **June 2019**

2074. *Proposed by Bao Do (student), Columbus State University, Columbus, GA.*

Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} H_k,$$

where $H_k = \sum_{j=1}^{k} \frac{1}{j}$ is the kth harmonic sum.

Solution by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule Mittelhessen, Friedberg, Germany.
Put

$$S_n := \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} H_k.$$

We have

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] H_k \\ &= S_{n-1} + \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} H_k \end{aligned}$$

and

$$\begin{aligned}
T_n &:= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} H_k \\
&= \sum_{k=1}^{n} (-1)^{k+1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] H_k \\
&= T_{n-1} + \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} H_{k+1} \\
&= T_{n-1} - T_{n-1} + \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{k+1} \\
&= \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \\
&= \frac{1}{n}.
\end{aligned}$$

The recursive formula $S_n = S_{n-1} + n^{-2}$ and $S_1 = 1$ imply $S_n = \sum_{k=1}^{n} k^{-2}$, therefore $\lim_{n\to\infty} S_n = \pi^2/6$.

Also solved by Michel Bataille (France), Khristo Boyadzhiev, Brian Bradie, David Bradly, Hongwei Chen, Robert Doucette, GWstat Problem Solving Group, Lixing Han, Tom Jager, Walther Janous (Austria), Dixon Jones, Albert Natian, José Nieto (Venezuela), Chikanna Selvaraj, Nicholas Singer, Albert Stadler (Switzerland), Seán Stewart (Australia), Michael Vowe (Switzerland), and the proposer. There was one incomplete or incorrect solution.

A recursively defined sequence of tetrahedra June 2019

2075. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore, MD and Mark Kaplan, Towson University, Towson, MD.*

Consider the sequence $\{C_n\}$ defined recursively by $C_0 = 3$, $C_1 = 1$, $C_2 = 3$, and

$$C_n = C_{n-1} + C_{n-2} + C_{n-3} \qquad \text{for } n \geq 3.$$

Let $O = (0, 0, 0)$ be the origin of $\mathbb{R}^3$ and, for integer $n \geq 0$, let P_n be the point (C_n, C_{n+1}, C_{n+2}).

(a) Find the volume of the pyramid $O P_n P_{n+1} P_{n+2}$ in closed form.
(b) Show that the sequence $\{P_n\}$ asymptotically approaches a fixed line $\mathcal{L}$ through the origin of $\mathbb{R}^3$, and characterize this line.

Solution by Brandon Cho (student), The Nueva School, San Mateo, CA.

(a) If we record the nonzero coordinates of tetrahedron $O P_n P_{n+1} P_{n+2}$ in the rows of the matrix

$$T_n = \begin{pmatrix} C_n & C_{n+1} & C_{n+2} \\ C_{n+1} & C_{n+2} & C_{n+3} \\ C_{n+2} & C_{n+3} & C_{n+4} \end{pmatrix},$$

then the recursion

$$T_0 = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 3 & 7 \\ 3 & 7 & 11 \end{pmatrix}, \quad T_{n+1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} T_n, \quad n \geq 0,$$

enables us to generate the nonzero coordinates of successive tetrahedra. Since

$$\det T_{n+1} = \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \det T_n = \det T_n,$$

$|\det T_n| = |\det T_0| = 44$ by induction. The volume of a tetrahedron formed by three vectors (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3) that are coterminal at the origin is

$$\frac{1}{6}\left|\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\right|.$$

Therefore, the volume of tetrahedron $OP_nP_{n+1}P_{n+2}$ is $\frac{1}{6}|\det T_n| = \frac{44}{6} = \frac{22}{3}$ for all $n \geq 0$.

(b) The characteristic equation for $\{C_n\}$ is

$$r^3 - r^2 - r - 1 = 0,$$

which has one real root $t \approx 1.839$ (the exact value can be determined by the cubic formula, but is not needed here) and two complex roots β and $\overline{\beta}$, each of whose modulus is $1/\sqrt{t} < 1$. Thus, for some constants k_1, k_2, and k_3,

$$C_n = k_1 t^n + k_2 \beta^n + k_3 \overline{\beta}^n, \quad n \geq 0.$$

We claim that $k_1 = k_2 = k_3 = 1$. To see this, let

$$D_n = t^n + \beta^n + \overline{\beta}^n.$$

Note that $D_0 = 3$ and by Vieta's relations

$$\begin{aligned} D_1 &= t + \beta + \overline{\beta} = 1 \text{ and} \\ D_2 &= \left(t + \beta + \overline{\beta}\right)^2 - 2\left(t\beta + t\overline{\beta} + \beta\overline{\beta}\right) \\ &= 1^2 - 2(-1) = 3. \end{aligned}$$

The initial conditions are satisfied, so $C_n = D_n$ for all $n \geq 0$.

Since

$$(C_n, C_{n+1}, C_{n+2}) - C_n(1, t, t^2) =$$
$$\left(0, \beta^n(\beta - t) + \overline{\beta}^n\left(\overline{\beta} - t\right), \beta^n\left(\beta^2 - t^2\right) + \overline{\beta}^n\left(\overline{\beta}^2 - t^2\right)\right)$$

and

$$\lim_{n\to\infty} \beta^n = \lim_{n\to\infty} \overline{\beta}^n = 0,$$

the P_n asymptotically approach the line through the origin with direction vector $(1, t, t^2)$.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Sarah Brickman (student), Robert Calcaterra, Robin Chapman (UK), George Washington University Problems Group, Eugene A. Herman, Tom Jager, Vitaliy Kushnirevych & Ulrich Abel (Germany), Harris Kwong, José H. Nieto (Venezuela), Jacob Siehler, Lawrence R. Weill, and the proposers. There were four incomplete or incorrect solutions.

Answers

Solutions to the Quickies from page 230.

A1101. The answer is yes.

Given a set

$$S = \{\alpha_1, \dots, \alpha_n\} \subset \mathbb{R},$$

we say β is *S-constructible* if we can construct a segment of length β from segments having lengths $\alpha_1, \dots, \alpha_n$ using only straightedge and compass. It is well known that if β_1 and β_2 are S-constructible, so are $\beta_1 + \beta_2$, $\beta_1\beta_2$, and β_1/n for any positive integer n. From these facts it is easy to see that $f(\alpha_1, \dots, \alpha_n)$ is S-constructible for any polynomial function f with rational coefficients. Also, if β is S-constructible, so is $\sqrt{\beta}$.

Denote the area of a polygon P by $A(P)$. Given an arbitrary polygon P, we want to show that $\sqrt{A(P)}$ is S-constructible, where S is the set of all side lengths and diagonal lengths of P. It is well known that P can be triangulated using diagonals. This gives

$$A(P) = \sum A(T_i),$$

where the T_i are the triangles in the triangulation of P. By Heron's formula

$$A(T_i) = \frac{\sqrt{2a_i^2b_i^2 + 2b_i^2c_i^2 + 2a_i^2c_i^2 - a_i^4 - b_i^4 - c_i^4}}{4},$$

where a_i, b_i, c_i are the sidelengths of T_i. Now $\{a_i, b_i, c_i\} \subseteq S$, so the $A(T_i)$ are S-constructible. Therefore $A(P)$ is S-constructible and finally $\sqrt{A(P)}$ is S-constructible.

Editor's Note. (i) The negative answer to the problem of duplicating the cube shows that one cannot "cube" an arbitrary polyhedron.

(ii) Similar arguments to those above show that using only straightedge and compass, one can construct a hypercube with the same 4-content as an arbitrary polytope.

A1102. We have

$$\begin{aligned} I_n &= -\int_0^1 \frac{1}{x^{n+1}} \sum_{i=n+1}^{\infty} \frac{x^i}{i}\, dx \\ &= -\int_0^1 \sum_{i=n+1}^{\infty} \frac{x^{i-n-1}}{i}\, dx \\ &\overset{(*)}{=} -\sum_{i=n+1}^{\infty} \frac{1}{i(i-n)} \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{\infty} \frac{1}{j(j+n)} \\
&= -\frac{1}{n} \sum_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{j+n} \right) \\
&= -\frac{1}{n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) \\
&= -\frac{H_n}{n}.
\end{aligned}
$$

Step (*) is justified since the power series has nonnegative terms.

Correction to Solution 2062

There is a typographical error in the solution to Problem 2062 in the February 2020 issue of THE MAGAZINE. The inequality in the last sentence should be $n > 10^{100}$. We thank Stan Wagon for pointing this out. He also notes that the *On-Line Encyclopedia of Integer Sequences* gives all 13 solutions to the problem:

28263827, 35000000, 242463827, 500000000, 528263827, 535000000, 10000000000, 10028263827, 10035000000, 10242463827, 10500000000, 10528263827, 10535000000

See http://oeis.org/A101639.