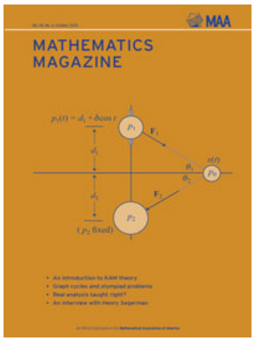

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Problems

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PROBLEMS

Proposals

To be considered for publication, solutions should be received by March 1, 2021.

2101. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore, MD and Mark Kaplan, Towson University, Towson, MD.*

Recall that the Steiner inellipse of a triangle is the unique ellipse that is tangent to each side of the triangle at the midpoints of those sides. Consider the Steiner inellipse E_S of $\triangle ABC$ and another ellipse, E_A, passing through the centroid G of $\triangle ABC$ and tangent to $\overleftrightarrow{AB}$ at B and to $\overleftrightarrow{AC}$ at C. If E_S and E_A meet at M and N, let $\angle MAN = \alpha$. Construct ellipses E_B and E_C, introduce their points of intersection with E_S, and define angles β and γ in an analogous way. Prove that

$$\frac{\cot\alpha + \cot\beta + \cot\gamma}{\cot A + \cot B + \cot C} = \frac{11}{3\sqrt{5}}.$$

2102. *Proposed by Donald Jay Moore, Wichita, KS.*

Let $\alpha = \pi/7$, $\beta = 2\pi/7$, and $\gamma = 4\pi/7$. Prove the following trigonometric identities.

$$\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\beta}{\cos^2\gamma} + \frac{\cos^2\gamma}{\cos^2\alpha} = 10,$$

$$\frac{\sin^2\alpha}{\sin^2\beta} + \frac{\sin^2\beta}{\sin^2\gamma} + \frac{\sin^2\gamma}{\sin^2\alpha} = 6,$$

$$\frac{\tan^2\alpha}{\tan^2\beta} + \frac{\tan^2\beta}{\tan^2\gamma} + \frac{\tan^2\gamma}{\tan^2\alpha} = 83.$$

Math. Mag. **93** (2020) 309–318. `doi:10.1080/0025570X.2020.1801039`. © Mathematical Association of America

We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at `http://mathematicsmagazine.submittable.com`. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by* LaTeX *source. General inquiries to the editors should be sent to* `mathmagproblems@maa.org`.

2103. *Proposed by Péter Kórus, University of Szeged, Szeged, Hungary.*

In a soccer game there are three possible outcomes: a win for the home team (denoted 1), a draw (denoted X), or a win for the visiting team (denoted 2). If there are n games, betting slips are printed for all 3^n possible outcomes. For four games, what is the minimum number of slips you must purchase to guarantee that at least three of the outcomes are correct on at least one of your slips?

2104. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

It is well known that no vector space can be written as the union of two proper subspaces. For which m does there exist a vector space V that can be written as a union of m proper subspaces with this collection of subspaces being minimal in the sense that no union of a proper subcollection is equal to V?

2105. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.*

Let $f : [0, 1] \to \mathbb{R}$ be a function that is k times differentiable on $[0, 1]$, with the kth derivative integrable on $[0, 1]$ and (left) continuous at 1. For integers $i \geq 1$ and $j \geq 0$ let

$$\sigma_j^{(i)} = \sum_{j_1+j_2+\cdots+j_i=j} 1^{j_1}2^{j_2}\cdots i^{j_i},$$

where the sum is extended over all i-tuples $(j_1, \ldots, j_i)$ of nonnegative integers that sum to j. Thus, for example, $\sigma_0^{(i)} = 1$, and $\sigma_1^{(i)} = 1 + 2 + \cdots + i = i(i + 1)/2$ for all $i \geq 1$. Also, for $0 \leq j \leq k$ let

$$a_j = \sigma_j^{(1)} f(1) + \sigma_{j-1}^{(2)} f'(1) + \cdots + \sigma_1^{(j)} f^{(j-1)}(1) + \sigma_0^{(j+1)} f^{(j)}(1).$$

Prove that

$$\int_0^1 x^n f(x)dx = \frac{a_0}{n} - \frac{a_1}{n^2} + \cdots + (-1)^k \frac{a_k}{n^{k+1}} + o\left(\frac{1}{n^{k+1}}\right),$$

for $n \to \infty$. As usual, we denote by $f^{(s)}$ the sth derivative of f (with $f^{(0)} = f$), and by $o(x_n)$ a sequence (y_n) with the property that $\lim_{n\to\infty} y_n/x_n = 0$.

Quickies

1103. *Proposed by Elias Lampakis, Kiparissia, Greece.*

Let a, b, and c be the side lengths of a triangle, r its inradius, and R its circumradius. Show that

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 108R^2r^2.$$

1104. *Proposed by George Stoica, Saint John, NB, Canada.*

Prove that, for every positive real number a, there exists a sequence $k_1, k_2, \ldots$ of positive integers such that $\{a \cdot 1^{k_1} \cdot 2^{k_2} \cdot \cdots \cdot n^{k_n}\} < 1/n$ for all $n \geq 1$. (Here $\{x\}$ denotes the fractional part of x.)

Solutions

Tribonacci and dual Tribonacci sequences **October 2019**

2076. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore, MD and Mark Kaplan, Towson University, Towson, MD.*

Given real numbers C_0, C_1, and C_2, one defines a *general Tribonacci (GT) sequence* $\{C_n\}$ recursively by the relation $C_{n+3} = C_{n+2} + C_{n+1} + C_n$ for all $n \geq 0$. Such GT-sequence $\{C_n\}$ is *nonsingular* if

$$\Delta = \begin{vmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \end{vmatrix} \neq 0.$$

A *dual Tribonacci (DT) sequence* $\{D_n\}$ is one that satisfies the dual recurrence $D_{n+3} + D_{n+2} + D_{n+1} = D_n$ for $n \geq 0$. Show that for any nonsingular GT-sequence $\{C_n\}$ with C_0, C_1, C_2 positive there exists a DT-sequence $\{D_n\}$ such that, for all $n \geq 0$,

$$\arctan\left(\frac{\sqrt{D_n}}{C_n}\right) = \arctan\left(\frac{\sqrt{D_n}}{C_{n+1}}\right) + \arctan\left(\frac{\sqrt{D_n}}{C_{n+2}}\right) + \arctan\left(\frac{\sqrt{D_n}}{C_{n+3}}\right).$$

Composite of solutions by Robert Calcaterra, University of Wisconsin-Platteville, Platteville, WI and Albert Stadler, Herrliberg, Switzerland.

Observe that the sequence where $D_n = 0$ for all n clearly satisfies the conditions of the problem. We will therefore endeavor to find a non-trivial solution to the problem.

The addition formula for the tangent function can be extended to show that

$$\tan(x + y + z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan x \tan z - \tan y \tan z}.$$

Therefore, the sequence D_n must satisfy

$$\frac{\sqrt{D_n}}{C_n} = \frac{\frac{\sqrt{D_n}}{C_{n+1}} + \frac{\sqrt{D_n}}{C_{n+2}} + \frac{\sqrt{D_n}}{C_{n+3}} - \frac{D_n^{3/2}}{C_{n+1}C_{n+2}C_{n+3}}}{1 - \frac{D_n}{C_{n+1}C_{n+2}} - \frac{D_n}{C_{n+1}C_{n+3}} - \frac{D_n}{C_{n+2}C_{n+3}}}. \tag{1}$$

If $D_n \neq 0$, we can divide both sides of the equation by $\sqrt{D_n}$ and solve for D_n to get

$$D_n = \frac{C_nC_{n+1}C_{n+2} + C_nC_{n+1}C_{n+3} + C_nC_{n+2}C_{n+3} - C_{n+1}C_{n+2}C_{n+3}}{C_n - C_{n+1} - C_{n+2} - C_{n+3}}.$$

Since C_n is a GT-sequence,

$$C_n - C_{n+1} - C_{n+2} - C_{n+3} = -2\left(C_{n+1} + C_{n+2}\right)$$

and

$$\begin{aligned} &C_nC_{n+2}C_{n+3} + C_nC_{n+1}C_{n+3} + C_nC_{n+1}C_{n+2} - C_{n+1}C_{n+2}C_{n+3} \\ &= C_nC_{n+2}C_{n+3} + C_nC_{n+1}C_{n+3} + C_{n+1}C_{n+2}(C_n - C_{n+3}) \\ &= C_nC_{n+2}C_{n+3} + C_nC_{n+1}C_{n+3} - C_{n+1}C_{n+2}(C_{n+1} + C_{n+2}) \\ &= (C_nC_{n+3} - C_{n+1}C_{n+2})(C_{n+1} + C_{n+2}). \end{aligned}$$

Therefore,

$$D_n = \frac{C_{n+1}C_{n+2} - C_nC_{n+3}}{2}.$$

Because all of the steps above are reversible, this does satisfy equation (1).

Since D_n may be negative, we recall that for real x

$$\arctan ix = \begin{cases} i \operatorname{arctanh}\ x & |x| < 1, \\ \operatorname{sign}(x)(\pi/2 + i \operatorname{arctanh}(1/|x|)) & |x| > 1. \end{cases}$$

Note that equation (1) may not translate into the desired statement about arctangents. For example, if $C_0 = 8$, $C_1 = 3$, $C_2 = 2$, then $C_3 = 13$ and $D_0 = -49$. We have

$$\arctan\left(\frac{\sqrt{D_0}}{C_0}\right) = \frac{\ln(15)}{2}i,$$

but

$$\arctan\left(\frac{\sqrt{D_0}}{C_1}\right) + \arctan\left(\frac{\sqrt{D_0}}{C_2}\right) + \arctan\left(\frac{\sqrt{D_0}}{C_3}\right) = \pi + \frac{\ln(15)}{2}i.$$

We will return to this issue shortly.

By the theory of linear recurrences, there are complex numbers a_1, a_2, a_3 such that

$$C_n = a_1r_1^n + a_2r_2^n + a_3r_3^n.$$

where r_1 is the real root and r_2, r_3 the conjugate complex roots of the equation $x^3 - x^2 - x - 1 = 0$. Note that by Vieta's formulas, $r_1r_2r_3 = 1$.

Now

$$\begin{aligned} 2D_n &= C_{n+1}C_{n+2} - C_nC_{n+3} \\ &= -a_1a_2(r_1 - r_2)^2(r_1 + r_2)r_3^{-n} - a_2a_3(r_2 - r_3)^2(r_2 + r_3)r_1^{-n} \\ &\quad - a_3a_1(r_3 - r_1)^2(r_3 + r_1)r_2^{-n}. \end{aligned}$$

Since D_n is a linear combination of powers of $1/r_1$, $1/r_2$, and $1/r_3$, it satisfies a linear recurrence whose characteristic polynomial has these values as roots. Therefore, the characteristic polynomial is $x^3 + x^2 + x - 1$ and D_n satisfies $D_{n+3} + D_{n+2} + D_{n+1} = D_n$ as desired.

The condition $\Delta \neq 0$ is equivalent to

$$a_1a_2a_3(r_1 - r_2)^2(r_2 - r_3)^2(r_3 - r_1)^2 \neq 0,$$

which means that $a_1, a_2, a_3 \neq 0$ and D_n is non-trivial. Finally, $|\sqrt{D_n}|$ grows like $1/\sqrt{|r_2|}^n$, while C_n grows like r_1^n, so $\sqrt{D_n}/C_n$ grows like $1/(r_1\sqrt{|r_2|})^n$. Since

$$r_1\sqrt{|r_2|} \approx 1.579, \lim_{n\to\infty} \sqrt{D_n}/C_n = 0$$

and for sufficiently large n we will be able to translate equation (1) into one involving arctangents.

Also solved by Elias Lampakis (Greece), Daniel Văcaru (Romania), and the proposers. There were two incomplete or incorrect solutions.

A triangle inequality **October 2019**

2077. *Proposed by Li Zhou, Polk State College, Winter Haven, FL.*

Prove that in any triangle with side lengths a, b, c, inradius r, and circumradius R, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{r}{R}>\frac{5}{3}.$$

Solution by Celia Schacht (graduate student), North Carolina State University, Raleigh, NC.

We first observe that three real numbers a, b, and c are the side lengths of a triangle if and only if there exist three positive real numbers x, y, and z, such that

$$a=x+y,\quad b=y+z,\quad \text{and}\quad c=x+z.$$

Without loss of generality, we may assume that $x \le y \le z$.

If s denotes the semiperimeter of the triangle and A its area, then in terms of x, y, and z, we have

$$s=x+y+z \;\text{ and }\; A=\sqrt{xyz(x+y+z)}.$$

It is well known that

$$r=\frac{A}{s} \;\text{ and }\; R=\frac{abc}{4A}.$$

Rewritten in term of x, y, and z, the original inequality becomes

$$\frac{x+y}{x+y+2z}+\frac{x+z}{x+2y+z}+\frac{y+z}{2x+y+z}+\frac{4xyz}{(x+y)(y+z)(x+z)}>\frac{5}{3}.$$

To prove this we will show that both

$$\frac{x+y}{x+y+2z}+\frac{x+z}{x+2y+z}>\frac{2}{3} \tag{1}$$

and

$$\frac{y+z}{2x+y+z}+\frac{4xyz}{(x+y)(y+z)(x+z)}\ge 1 \tag{2}$$

hold.

Now (1) is equivalent to

$$3(x+y)(x+2y+z)+3(x+z)(x+y+2z)>2(x+y+2z)(x+2y+z)$$

which can be rewritten as

$$2x^2+3xy+3xz+(y-z)^2>0.$$

Since x, y, and z are positive, this holds.

Note that (2) is equivalent to

$$(y+z)^2(x+z)(x+y)+4xyz(2x+y+z)\ge(2x+y+z)(y+z)(x+z)(x+y),$$

which (since $x>0$) is equivalent to

$$2xyz+y^2z+yz^2\ge x^2y+xy^2+x^2z+xz^2.$$

This can be easily seen to be true, by noticing that, since $x \le y \le z$, we have

$$xyz \ge x^2z, xyz \ge xy^2, yz^2 \ge xz^2, y^2z \ge x^2y.$$

The original inequality is sharp, as one can see by putting $a = c$, $b = 2a - \epsilon$, and letting $\epsilon \to 0$.

Also solved by Michel Bataille (France), Robert Calcaterra, Habib Far, Subhankar Gayen (India), Michael Goldenberg & Mark Kaplan, Walther Janous (Austria), Omran Kouba (Syria), Elias Lampakis (Greece), Kee-Wai Lau (Hong Kong), Volkhard Schindler (Germany), Albert Stadler (Switzerland), Daniel Văcaru (Romania), Michael Vowe (Switzerland), John Zacharias and the proposer.

A nilpotent commutator **October 2019**

2078. *Proposed by Florin Stanescu, Şerban Cioculescu School, Găeşti, Romania.*

Let A, B be $n \times n$ complex matrices such that $A^2 + B^2 = 2AB$. Prove that $(AB - BA)^m = \mathbf{0}$ for some $m \le \lceil \frac{n}{2} \rceil$.

Solution by Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong.
Let $X = A - B$ and rewrite the given condition as $X^2 = AB - BA$. We first show that X is nilpotent. Rewriting the given condition as $X^2 = XB - BX$, and multiplying both sides by X^{k-1} on the right gives

$$X^{k+1} = XBX^{k-1} - BX^k.$$

Thus

$$\operatorname{tr}(X^{k+1}) = \operatorname{tr}(XBX^{k-1}) - \operatorname{tr}(BX^k) = \operatorname{tr}(BX^{k-1} \cdot X) - \operatorname{tr}(BX^k) = 0$$

for all $k \ge 1$.

If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of X, then

$$\operatorname{tr}(X^\ell) = \sum_{i=1}^{n} \lambda_i^\ell = 0$$

for all $\ell \ge 2$. This forces $\lambda_i = 0$ for all i, so the characteristic polynomial for X is λ^n. By the Cayley–Hamilton theorem, $X^n = \mathbf{0}$.

Let $m = \lceil n/2 \rceil$. Then $(AB - BA)^m = X^{2m} = \mathbf{0}$ since $2m \ge n$ and we are done.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Eugene Herman, John C. Kieffer, Julio Cesar Mohnsam (Brazil), Northwestern University Problem Solving Group, Daniel Văcaru (Romania), and the proposer. There was one incomplete or incorrect solution.

An improper integral that almost never converges **October 2019**

2079. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Given real numbers a, b with $b > 0$, prove that the integral

$$J(a, b) := \int_0^\infty \left[2 + (x + a) \ln\left(\frac{x}{x + b}\right)\right] dx$$

converges if and only if $a = 1$ and $b = 2$, and find the value $J(1, 2)$.

Solution by Eugene Herman, Grinnell, IA.
Let $F(x)$ denote an antiderivative of the integrand. Integrate by parts using the antiderivatives $x(x+2a)/2$ and $(x+2a-b)(x+b)/2$ of $x+a$:

$$\begin{aligned}F(x) &= 2x + \int (x+a)\ln x\,dx - \int (x+a)\ln(x+b)\,dx \\ &= 2x + \frac{1}{2}x(x+2a)\ln x - \int \frac{1}{2}(x+2a)\,dx \\ &\quad - \frac{1}{2}(x+2a-b)(x+b)\ln(x+b) + \int \frac{1}{2}(x+2a-b)\,dx \\ &= 2x + \frac{1}{2}x(x+2a)\ln x - \frac{1}{2}(x+2a-b)(x+b)\left(\ln x - \ln\left(\frac{x}{x+b}\right)\right) - \frac{bx}{2}\end{aligned}$$

Note that $\lim_{x\to 0^+} F(x) = -(2a-b)b(\ln b)/2$. So convergence of the integral depends entirely on convergence of $\lim_{x\to\infty} F(x)$. We use

$$\ln(1-t) = -t - \frac{1}{2}t^2 + O(t^3) \quad \text{as } t \to 0^+$$

and so

$$\ln\left(\frac{x}{x+b}\right) = \ln\left(1 - \frac{b}{x+b}\right) = -\frac{b}{x+b} - \frac{1}{2}\left(\frac{b}{x+b}\right)^2 + O(1/x^3) \quad \text{as } x \to \infty$$

Hence

$$\begin{aligned}F(x) &= 2x - \frac{bx}{2} + \frac{b^2-2ab}{2}\ln x + \frac{x+2a-b}{2}(x+b)\left(-\frac{b}{x+b} - \frac{1}{2}\left(\frac{b}{x+b}\right)^2\right) + O\left(\frac{1}{x}\right) \\ &= 2x - \frac{bx}{2} + \frac{b^2-2ab}{2}\ln x - \frac{(x+2a-b)b}{2} - \frac{(x+2a-b)b^2}{4(x+b)} + O\left(\frac{1}{x}\right)\end{aligned}$$

In order that $\lim_{x\to\infty} F(x)$ converge, the coefficients of x and $\ln x$ must be zero. Thus, $2 - b/2 - b/2 = 0$ and $b(b-2a)/2 = 0$, and so $b = 2$ and $a = 1$. Therefore, since $\lim_{x\to 0^+} F(x) = 0$,

$$J(1,2) = \lim_{x\to\infty} -\frac{4x}{4(x+2)} = -1$$

Also solved by Ulrich Abel (Germany), Michel Bataille (France), Paul Bracken, Brian Bradie, Robert Calcaterra, Hongwei Chen, Paul Deiermann, Shuyang Gao, Finbarr Holland (Ireland), Walther Janous (Austria), Elias Lampakis (Greece), Missouri State University Problem Solving Group, Angel Plaza (Spain), Arthur Rosenthal, Albert Stadler (Switzerland), Daniel Văcaru (Romania), and the proposers. There were three incomplete or incorrect solution.

Edge colorings with no monochromatic triangles **October 2019**

2080. *Proposed by the UTSA Problem Solving Club, University of Texas at San Antonio, San Antonio, TX.*

For $n \geq 3$, let W_n be the wheel graph consisting of an n-cycle all whose vertices are joined to an additional distinct vertex.

(i) How many colorings of the $2n$ edges of W_n using $k \geq 2$ colors result in no monochromatic triangles?

(ii) Regard two colorings of W_n as equivalent if there is a graph automorphism of W_n that maps the first coloring to the second. If $k \geq 2$ and $p > 3$ is prime, count all non-equivalent colorings of W_p using k colors.

Solution by Rob Pratt, Apex, NC.

(i) We apply the principle of inclusion and exclusion (PIE). For $n = 3$, there are six edges and four triangles. Ignoring monochromaticity, there are k^6 edge colorings. For each monochromatic triangle, there are k ways to color the triangle and k^{6-3} ways to color the remaining edges. For each pair of monochromatic triangles (which must share an edge), there are k ways to color the triangles and k ways to color the remaining edge. For three (equivalently, four) monochromatic triangles, there are k ways to color the triangles and no remaining edges. So PIE yields

$$k^6 - \binom{4}{1}k \cdot k^{6-3} + \binom{4}{2}k \cdot k - \binom{4}{3}k + \binom{4}{4}k = k^6 - 4k^4 + 6k^2 - 3k$$

edge colorings with no monochromatic triangles.

For $n > 3$, there are n triangles. For $t < n$ monochromatic triangles with s shared edges, there are k^{t-s} ways to color the triangles and $k^{2n-3t+s}$ ways to color the remaining edges. If all n triangles are monochromatic, there are k colorings. Now PIE yields

$$\begin{aligned}
&\sum_{t=0}^{n-1}(-1)^t\binom{n}{t}k^{t-s} \cdot k^{2n-3t+s} + (-1)^n k \\
&= \sum_{t=0}^{n}(-1)^t\binom{n}{t}(k^2)^{n-t} - (-1)^n\binom{n}{n}(k^2)^{n-n} + (-1)^n k \\
&= (k^2-1)^n + (-1)^n(k-1).
\end{aligned}$$

(ii) We apply the Cauchy–Frobenius–Burnside theorem, which states that the number of equivalence classes of a finite set X under the action of a finite group G is

$$\frac{1}{|G|}\sum_{g \in G} F_g,$$

where $F_g = |\{x \in X : g \cdot x = x\}|$ is the cardinality of the set of fixed points of $g \in G$.

The automorphism group of the wheel graph W_p is the dihedral group of order $2p$. The identity element fixes all k^{2p} edge colorings. Because $p > 3$ is prime, each nontrivial rotation fixes only the colorings for which both the p spokes and the sides of the outer p-cycle are monochromatic, yielding k^2 colorings. Each reflection fixes only the colorings for which the $(2p-2)/2 = p-1$ edge pairs across the reflection are monochromatic, and the two self-reflective edges can take any color. Hence the number of non-equivalent colorings is

$$\frac{1}{2p}\left(k^{2p} + (p-1)k^2 + p \cdot k^2 \cdot k^{p-1}\right) = \frac{k^{2p} + (p-1)k^2 + p \cdot k^{p+1}}{2p}.$$

Editor's Note. The proposers' intent was that part (ii) also required triangles to be non-monochromatic, but this was not explicitly stated. Here is a solution to the intended problem by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.

(ii) Denote the central vertex by v_0 and the vertices of the n-cycle by $v_1, \ldots, v_n$. Let r_i denote the spoke $v_0 v_i$ and call the edges $v_1 v_2, \ldots, v_n v_1$ sides.

The identity leaves fixed all the colorings, namely $(k^2-1)^p - k + 1$ of them by part (i).

Since p is prime, any of the $p-1$ rotations other than the identity is a generator of the group of rotations, hence for a coloring to be fixed, all the spokes must be of the same color and also all the sides must have the same color (different from the color of the spokes). Thus there are $k(k-1)$ such colorings.

Let $m = (p+1)/2$. There are $\binom{m-1}{j}$ ways to choose exactly j pairs of consecutive spokes from $r_1, \ldots, r_m$ with the same color. This gives $m-j$ clusters of adjacent spokes each cluster having the same color. There are $k(k-1)^{m-j-1}$ ways to color these clusters. This gives a total of $\binom{m-1}{j}k(k-1)^{m-1-j}$ ways to color $r_1, \ldots, r_m$ with exactly j pairs of consecutive spokes having the same color. Now consider a reflection, say the one through v_1. Each of the spoke colorings can be extended to a coloring of W_p without monochromatic triangles and invariant under this reflection in $(k-1)^{j+1}k^{m-1-j}$ ways (the exponent of $(k-1)$ is $j+1$ to account for the color of the side $v_m v_{m+1}$). Hence the number of colorings invariant under this reflection is

$$\sum_{j=0}^{m-1} \binom{m-1}{j} k^{m-j}(k-1)^m = k(k-1)^m(k+1)^{m-1} = k(k-1)^{\frac{p+1}{2}}(k+1)^{\frac{p-1}{2}}.$$

Finally the number of non-equivalent colorings is

$$= \frac{1}{2p}\left((k^2-1)^p - k + 1 + (p-1)k(k-1) + pk(k-1)^{\frac{p+1}{2}}(k+1)^{\frac{p-1}{2}}\right)$$

$$= \frac{1}{2}k(k-1)\left(1 + (k^2-1)^{\frac{p-1}{2}}\right) + \frac{1}{2p}(k^2-1)\left((k^2-1)^{p-1} - 1\right).$$

We note that the case $p = 3$ may be handled with the same techniques. The group of automorphisms is isomorphic to S_4 and the number of non-equivalent colorings is

$$\frac{1}{24}k^3(k-1)(k^3+k^2+6k-6).$$

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), and the proposers.

Answers

Solutions to the Quickies from page 310.

A1103. It is well known that

$$r = \frac{2A}{a+b+c} \quad \text{and} \quad R = \frac{abc}{4A},$$

where A denotes the area of the triangle. Therefore the inequality we wish to prove is equivalent to

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 27\frac{a^2b^2c^2}{(a+b+c)^2}$$

or equivalently

$$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(a+b+c)^2 \geq 27.$$

Hölder's inequality is

$$\left(\sum_{i=1}^{k} x_i^p\right)^{1/p}\left(\sum_{i=1}^{k} y_i^q\right)^{1/q} \geq \sum_{i=1}^{k} x_i y_i,$$

where $x_i, y_i \geq 0$ and $1/p + 1/q = 1$. Let $k = 3$, $p = 3$, $q = 3/2$, and

$$x_1 = \frac{1}{a^{2/3}}, x_2 = \frac{1}{b^{2/3}}, x_3 = \frac{1}{c^{2/3}}, y_1 = a^{2/3}, y_2 = b^{2/3}, y_3 = c^{2/3}.$$

Then Hölder's inequality becomes

$$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^{1/3}(a+b+c)^{2/3} \geq 3$$

or equivalently

$$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(a+b+c)^2 \geq 27,$$

which is what we wanted to show. Note that the case when $a = b = c$ shows that this inequality is sharp.

A1104. Let us denote by a_n the fractional part of $a \cdot 1^{k_1} \cdot 2^{k_2} \cdot \cdots \cdot n^{k_n}$, $n \geq 1$.

Put $k_1 = 1$ and assume that we have already defined $k_2, \ldots, k_n$ such that $a_1 < 1$, $a_2 < 1/2, \ldots, a_n < 1/n$. Keeping in mind that

$$\frac{1}{n} = \sum_{i=1}^{\infty} \frac{1}{(n+1)^i},$$

we either have that

$$a_n < \frac{1}{n+1}, \tag{1}$$

or there exists a unique positive integer $p \geq 1$ such that

$$\frac{1}{n+1} + \cdots + \frac{1}{(n+1)^p} \leq a_n < \frac{1}{n+1} + \cdots + \frac{1}{(n+1)^{p+1}}. \tag{2}$$

In the first case, put $k_{n+1} = 0$; in the second case put $k_{n+1} = p$.

In both cases, a_{n+1}, which is the fractional part of the product $a_n \cdot (n+1)^{k_{n+1}}$, is less than $\frac{1}{n+1}$, by either (1) or (2), and the induction process is complete.