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Problems and Solutions

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PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman**, and **Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Stan Wagon, Lawrence Washington, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
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Proposed solutions to the problems below should be submitted by December 31, 2020, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

We reprint problem 12185 from the May 2020 issue, correcting errors.

12185. *Proposed by George Stoica, Saint John, NB, Canada.* Let $n_1, \dots, n_k$ be pairwise relatively prime odd integers greater than 1. For $i \in \{1, \dots, k\}$, let $f_i(x) = \sum_{m=1}^{n_i} x^{m-1}$. Let A be a $2k$-by-$2k$ matrix with real entries such that $\det f_j(A) = 0$ for all $j \in \{1, \dots, k\}$. Prove $\det A = 1$.

12195. *Proposed by Joseph DeVincentis, Salem, MA, James Tilley, Bedford Corners, NY, and Stan Wagon, Macalester College, St. Paul, MN.* For which integers n with $n \geq 3$ can a regular n-gon be inscribed in a cube? The vertices of the n-gon must all lie on the cube but may not all lie on a single face.

12196. *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.* Determine which positive integers n have the following property: If $a_1, \dots, a_n$ are n real numbers greater than or equal to 1, and A, G, and H are their arithmetic mean, geometric mean, and harmonic mean, respectively, then

$$G - H \geq \frac{1}{G} - \frac{1}{A}.$$

12197. *Proposed by Nicolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Prove that the equation

$$(a^2 + 1)(b^2 - 1) = c^2 + 3333$$

has no solutions in integers a, b, and c.

12198. *Proposed by Michel Bataille, Rouen, France.* Let $A_1A_2A_3$ be a nonequilateral triangle with incenter I, circumcenter O, and circumradius R. For $i \in \{1, 2, 3\}$, let B_i be the point of tangency of the incircle of $A_1A_2A_3$ with the side of the triangle opposite A_i, and let C_i be the point of intersection between the circle centered at I of radius R and the ray IB_i. Let K be the orthocenter of $C_1C_2C_3$. Prove that I is the midpoint of OK.

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12199. *Proposed by Shivam Sharma, Delhi University, New Delhi, India.* Prove

$$\int_0^\infty \frac{x\sinh(x)}{3+4\sinh^2(x)}\,dx = \frac{\pi^2}{24}.$$

12200. *Proposed by Ibrahim Suat Evren, Denizli, Turkey.* Prove that for every positive integer m, there is a positive integer k such that k does not divide $m+x^2+y^2$ for any positive integers x and y.

12201. *Proposed by Stephen M. Gagola, Jr., Kent State University, Kent, OH.* Let F be a field, and let G be a finite group. The *group algebra* $F[G]$ is the vector space of all formal sums $\sum_{g\in G} a_g g$, where $a_g \in F$, with multiplication defined by extending the multiplication in G via the distributive laws. A subset S of $F[G]$ is G*-invariant* if $s \in S$ and $g \in G$ imply $sg \in S$. In particular, the subset G is G-invariant, as is the singleton set $\{\sum_{g\in G} g\}$. Find all fields F and groups G such that there exists an F-linear transformation $\phi\colon F[G] \to F[G]$ that is not right multiplication by an element of G but that nevertheless sends every G-invariant subset to itself.

SOLUTIONS

An Inequality on Means

12083 [2019, 82]. *Proposed by Alijadallah Belabess, Khemisset, Morocco.* Let x, y, and z be positive real numbers. Prove

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \ge \frac{3\sqrt{3}}{2\sqrt{x^2+y^2+z^2}}.$$

Solution by Peter W. Lindstrom, Saint Anselm College, Manchester, NH. Applying the arithmetic mean–harmonic mean inequality to $x+y$, $y+z$, and $z+x$ gives

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \ge \frac{9}{(x+y)+(y+z)+(z+x)} = \frac{9}{2(x+y+z)}.$$

The Cauchy–Schwarz inequality yields

$$x+y+z \le \sqrt{3}\sqrt{x^2+y^2+z^2}.$$

Thus

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \ge \frac{9}{2(x+y+z)} \ge \frac{3\sqrt{3}}{2\sqrt{x^2+y^2+z^2}}.$$

Also solved by M. Aassila (France), R. A. Agnew, S. Amghibech (Canada), F. R. Ataev (Uzbekistan), D. Bailey & E. Campbell & C. Diminnie & T. Smith, M. Bataille (France), A. Berkane (Algeria), P. Bracken, B. Bradie, R. Chapman (UK), H. Chen, W. J. Cowieson, P. P. Dályay (Hungary), H. L. Das, P. De (India), A. B. Dixit (Canada), H. Y. Far, G. Fera (Italy), D. Fleischman, A. Garcia (France), S. Gayen (India), M. Getz & D. Jones, O. Geupel (Germany), N. Grivaux (France), J. Grzesik, E. A. Herman, E. J. Ionaşcu, P. Ivády (Hungary), W. Janous (Austria), N. R. Johnson, M. Goldenberg & M. Kaplan, B. Karivanov (USA) & T. S. Vassilev (Canada), P. Khalili, K. T. L. Koo (China), O. Kouba (Syria), S. S. Kumar, H. Kwong, W.-K. Lai & J. Risher, K.-W. Lau (China), Y. E. Lee, J. H. Lindsey II, L. Lipták, O. P. Lossers (Netherlands), A. Mahillo (Spain), D. Ş. Marinescu & M. Mihai (Romania), L. Matejička (Slovakia), S. Meherrem (Turkey), V. Mikayelyan (Armenia), R. Molinari, R. Nandan, H. L. Nhat, P. Nüesch (Switzerland), H. Ohtsuka (Japan), A. Pathak, A. D. Pirvuceanu (Romania), Á. Plaza (Spain), C. R. Pranesachar (India), E. Rajabli (Azerbaijan), M. Reid,

H. Ricardo, D. Ritter, R. E. Rogers, C. Schacht, K. Schilling, S. B. Seales, R. Shinde (India), T. de Souza Leão, A. Stadler (Switzerland), N. Stanciu & T. Zvonaru (Romania), R. Stong, R. Tauraso (Italy), T. Toyonari (Japan), D. B. Tyler, E. I. Verriest, J. Vinuesa (Spain), M. Vowe (Switzerland), E. A. Weinstein, T. Wiandt, H. Widmer (Switzerland), M. R. Yegan (Iran), J. Zacharias, L. Zhou, GCHQ Problem Solving Group (UK), Iowa State University Problem Solving Group, and the proposer.

A Characterization of Unbounded Averages

12084 [2019, 82]. *Proposed by George Stoica, Saint John, NB, Canada.* Let $a_1, a_2, \ldots$ be a sequence of nonnegative numbers. Prove that $(1/n)\sum_{k=1}^{n} a_k$ is unbounded if and only if there exists a decreasing sequence $b_1, b_2, \ldots$ such that $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n$ is finite, and $\sum_{n=1}^{\infty} a_n b_n$ is infinite. Is the word "decreasing" essential?

Solution by Gérard Lavau, Fontaine lès Dijon, France. Let $S_0 = 0$, and for $n > 0$ let $S_n = \sum_{k=1}^{n} a_k$.

(1) Let $b_1, b_2, \ldots$ be a decreasing sequence such that $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n$ is finite, and $\sum_{n=1}^{\infty} a_n b_n$ is infinite. Clearly $b_n \ge 0$ for all n. If $(1/n)\sum_{k=1}^{n} a_k$ is bounded, say by M, then $S_n \le Mn$ for all n. By Abel summation by parts,

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (S_k - S_{k-1}) b_k = S_n b_n + \sum_{k=1}^{n-1} S_k (b_k - b_{k+1})$$

$$\le Mnb_n + \sum_{k=1}^{n-1} Mk(b_k - b_{k+1}) = M\sum_{k=1}^{n} b_k \le M\sum_{k=1}^{\infty} b_k,$$

which contradicts our assumption that $\sum_{k=1}^{\infty} a_n b_n$ is infinite.

(2) Conversely, suppose $(1/n)\sum_{k=1}^{n} a_k$ is unbounded. For all M, we have $S_n > Mn$ for sufficiently large n. Let $n_0 = 0$, and for $k > 0$ choose n_k so that $n_k > n_{k-1}$ and $S_{n_k} > 4^k n_k$. Let $b_j = 1/(2^k n_k)$ for $n_{k-1} + 1 \le j \le n_k$. Clearly the sequence $b_1 . b_2 . \ldots$ is decreasing and tends to 0. For all $k > 0$ we have

$$\sum_{j=1}^{n_k} b_j = \sum_{i=1}^{k} \sum_{j=n_{i-1}+1}^{n_i} b_j = \sum_{i=1}^{k} \frac{n_i - n_{i-1}}{2^i n_i} \le \sum_{i=1}^{k} \frac{1}{2^i} \le \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.$$

Thus, $\sum_{j=1}^{\infty} b_j$ is finite. On the other hand,

$$\sum_{j=1}^{n_k} a_j b_j \ge \sum_{j=1}^{n_k} a_j b_{n_k} = \frac{1}{2^k n_k} \sum_{j=1}^{n_k} a_j = \frac{S_{n_k}}{2^k n_k} \ge \frac{4^k n_k}{2^k n_k} = 2^k,$$

so $\sum_{j=1}^{\infty} a_j b_j$ is infinite.

If the word "decreasing" is removed, then part (1) may fail. For $n \ge 1$ let $b_n = (-1)^n/\sqrt{n}$, $a_{2n} = 1/\sqrt{n}$, and $a_{2n-1} = 0$. The series $\sum_{n=1}^{\infty} b_n$ converges by the alternating series test, but

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_{2n} b_{2n} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2}},$$

and so $\sum_{n=1}^{\infty} a_n b_n$ diverges. Nevertheless, $(1/n)\sum_{k=1}^{n} a_k$ is bounded, since

$$\frac{1}{n}\sum_{k=1}^{n} a_k \le \frac{1}{n}\sum_{k=1}^{2n} a_k = \frac{1}{n}\sum_{k=1}^{n} a_{2k} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \le \frac{1}{n}\int_0^n \frac{dx}{\sqrt{x}} = \frac{2\sqrt{n}}{n} \le 2.$$

Editorial comment. Most solutions used summation by parts or the transformation $b_n = \sum_{k=n}^{\infty} x_k/k$. In (2), the sequence $b_1, b_2, \dots$ can be made strictly decreasing by replacing it with the sequence $b_1', b_2', \dots$, where $b_k' = b_k + 1/2^k$. When the word "decreasing" is removed, one can construct a counterexample to (1) with nonnegative b_k as follows: $a_k = k$ and $b_k = 1/k$ when k is a power of 2, and $a_k = b_k = 0$ otherwise.

Also solved by K. F. Andersen (Canada), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), H. Park, K. Schilling, R. Stong, R. Tauraso (Italy), and the proposer.

Subsets with Equal Sums of Like Powers, Revisited

12085 [2019, 82]. *Proposed by Joseph DeVincentis, Salem, MA, Stan Wagon, Macalester College, St. Paul, MN, and Michael Elgersma, Plymouth, MN.* For which positive integers n can $\{1, \dots, n\}$ be partitioned into two sets A and B of the same size so that

$$\sum_{k\in A} k = \sum_{k\in B} k, \quad \sum_{k\in A} k^2 = \sum_{k\in B} k^2, \quad \text{and} \quad \sum_{k\in A} k^3 = \sum_{k\in B} k^3?$$

Solution by Greg Marks, St. Louis University, St. Louis, MO. These conditions hold precisely for $n \in \{8k\colon\ k \ge 2\}$.

First we show that divisibility by 8 is necessary. The condition $|A| = |B|$ requires $2 \mid n$. The condition $\sum_{k\in A} k = \sum_{k\in B} k = n(n+1)/4$ then requires $4 \mid n$. Let $m = n/4$. The conditions yield $\sum_{k\in A} k^2 = \frac{1}{2}\sum_{k=1}^{n} k^2 = m(4m+1)(8m+1)/3$ and $\sum_{k\in A} k^3 = \frac{1}{2}\sum_{k=1}^{n} k^3 = 2m^2(4m+1)^2$. When m is odd, these two sums have opposite parity. However, $\sum_{k\in A}(k^3 - k^2)$ is even, since each summand is even. Therefore, $8 \mid n$.

We write $[n]$ for the set $\{1, \dots, n\}$. Let $\mathcal{P}_\ell$ denote the set of positive integers n such that $[n]$ decomposes as the disjoint union of sets A and B satisfying $\sum_{k\in A} k^i = \sum_{k\in B} k^i$ for $0 \le i \le \ell$. We show that $\mathcal{P}_\ell$ is an additive semigroup in $\mathbb{N}$. Given $n_1, n_2 \in \mathcal{P}_\ell$, let $\{A_1, B_1\}$ and $\{A_2, B_2\}$ be such decompositions of $[n_1]$ and $[n_2]$, respectively. Let $A = A_1 \cup \{a + n_1\colon\ a \in A_2\}$ and $B = B_1 \cup \{b + n_1\colon\ b \in B_2\}$. Now A and B decompose $[n]$, and the binomial theorem yields a proof by induction on i that $\sum_{k\in A} k^i = \sum_{k\in B} k^i$ for $0 \le i \le \ell$. Hence $n_1 + n_2 \in \mathcal{P}_\ell$.

Since $\{16, 24\}$ generates $\{8k\colon\ k \ge 2\} \subset \mathbb{N}$ as an additive semigroup, to prove $\mathcal{P}_3 = \{8k\colon\ k \ge 2\}$ it suffices to show that $16, 24 \in \mathcal{P}_3$ and $8 \notin \mathcal{P}_3$.

To avoid a brute-force search, the key observation is that $n \in \mathcal{P}_\ell$ implies $2n \in \mathcal{P}_{\ell+1}$. Given a decomposition $\{A, B\}$ of $[n]$ witnessing $n \in \mathcal{P}_\ell$, let $A' = A \cup \{b + n\colon\ b \in B\}$ and $B' = B \cup \{a + n\colon\ a \in A\}$. Now $\{A', B'\}$ is a decomposition of $[2n]$, and another application of the binomial theorem yields $\sum_{k\in A} k^{\ell+1} = \sum_{k\in B} k^{\ell+1}$. The earlier computation yields $2n \in \mathcal{P}_\ell$, so now also $2n \in \mathcal{P}_{\ell+1}$.

Decomposing $[4]$ into $\{1, 4\}$ and $\{2, 3\}$ yields $4 \in \mathcal{P}_1$; hence $8 \in \mathcal{P}_2$ and $16 \in \mathcal{P}_3$. Decomposing $[12]$ into $\{1, 3, 7, 8, 9, 11\}$ and $\{2, 4, 5, 6, 10, 12\}$ yields $12 \in \mathcal{P}_2$. Hence $24 \in \mathcal{P}_3$.

Finally, consider $[8]$. The only partition of $[8]$ that establishes $8 \in \mathcal{P}_2$ is the pair of sets $\{1, 4, 6, 7\}$ and $\{2, 3, 5, 8\}$. Since $1^3 + 4^3 + 6^3 + 7^3 \ne 2^3 + 3^3 + 5^3 + 8^3$, it follows that $8 \notin \mathcal{P}_3$. Thus $\mathcal{P}_3 = \{8k\colon\ k \ge 2\}$.

Editorial comment. Rory Molinari pointed out that $\{8k\colon\ k \ge 2\} \subseteq \mathcal{P}_3$ was proved by Tarry and Barbette; see Theorem 7 of H. L. Dorwart and O. E. Brown (1937), The Tarry–Escott problem, this MONTHLY 44(10): 613–626. O. P. Lossers mentioned the relation with the Thue–Morse sequence and the Prouhet–Tarry–Escott problem.

As recounted by E. M. Wright (1959), Prouhet's 1851 solution of the Tarry–Escott problem of 1910, this MONTHLY 66(3): 199–201, Tarry and Escott independently in 1912 and 1910 gave generalizations of the fact that the integers $0, \dots, 2^{k+1} - 1$ can be split into two sets A and B so that the sum of the mth powers of elements of the two sets are equal for $0 \le m \le k$. However, Prouhet had stated this more generally for j sets. Within $\{0, \dots, j^{k+1} - 1\}$, let A_i consist of those numbers whose entries in their j-ary expansions have sum congruent to i modulo j. The result is that the sum of the mth powers of the numbers in A_i is independent of i, for each m with $0 \le m \le k$. In addition to a proof in the cited article, other proofs appear in D. H. Lehmer (1947), The Tarry–Escott problem, *Scripta Math.* 13: 37–41, and E. M. Wright (1949), Equal sums of like powers, *Proc. Edinburgh Math. Soc.* (2)8: 138–142.

The Tarry–Escott Problem appeared again as recently as Problem 10284 [1993, 185; 1995, 843] in this MONTHLY. The problem has a substantial literature, including a short book: A. Gloden (1944), *Mehrgradige Gleichungen*, 2nd ed., Groningen: P. Noordhoff.

For the problem of determining the set $\mathcal{P}_k$ of integers for which such splittings occur, the case $k = 3$ considered here was solved in D. W. Boyd (1997), On a problem of Byrnes concerning polynomials with restricted coefficients, *Math. Comp.* 66: 1697–1703. The set $\mathcal{P}_k$ is now known for k up to 7; see J. Buhler, S. Golan, R. Pratt, and S. Wagon (2019), Symmetric Littlewood polynomials, spectral-null codes, and equipowerful partitions, arxiv.org/abs/1912.03491.

Also solved by K. David & A. van Groningen, S. M. Gagola Jr., K. Gatesman, Y. J. Ionin, M. E. Kidwell & M. D. Meyerson, P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Molinari, M. Reid, N. C. Singer, R. Tauraso (Italy), M. Wildon, GCHQ Problem Solving Group (UK), Missouri State University Problem Solving Group, and the proposers.

Maximizing the Area of an Incenter Triangle

12086 [2019, 82]. *Proposed by Miguel Ochoa Sanchez, Lima, Peru, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let ABC be a triangle with right angle at A, and let H be the foot of the altitude from A. Let M, N, and P be the incenters of triangles ABH, ABC, and ACH, respectively. Prove that the ratio of the area of triangle MNP to the area of triangle ABC is at most $(\sqrt{2} - 1)^3/2$, and determine when equality holds.

Solution by Dmitry Fleischman, Santa Monica, CA. Let the sides of the triangle be denoted a, b, and c, as usual, and let the inradii of $\triangle ABC$, $\triangle ACH$, and $\triangle ABH$ be denoted r_A, r_B, and r_C, respectively. As is well known, the altitude on the hypotenuse of a right triangle divides the triangle into two smaller triangles that are similar to it. All corresponding sides, as well as any other corresponding linear measurements such as altitudes and inradii, are in the proportion $a : b : c$, which are the hypotenuse lengths of the three triangles. In particular, the inradii r_A, r_B, and r_C are in the proportion $a : b : c$.

Let $K(XYZ)$ denote the area of $\triangle XYZ$. We determine $K(MNP)/K(ABC)$ by computing the two ratios $K(MNP)/K(BNC)$ and $K(BNC)/K(ABC)$.

The angle bisector at B contains both M and N, and likewise the angle bisector at C contains both P and N. Hence B, M, and N are collinear, as are C, P, and N. Let the projections of M, N, and P onto BC be denoted M', N', and P', respectively. Since $\triangle MNP$ and $\triangle BNC$ share the angle at N, the ratio of their areas is the product of NM/NB and NP/NC. By similar triangles,

$$\frac{NM}{NB} = 1 - \frac{MB}{NB} = 1 - \frac{MM'}{NN'} = 1 - \frac{r_C}{r_A} = 1 - \frac{c}{a},$$

and a similar calculation shows that $NP/NC = 1 - b/a$. Thus

$$\frac{K(MNP)}{K(BNC)} = \left(1 - \frac{b}{a}\right)\left(1 - \frac{c}{a}\right) = 1 - \frac{b+c}{a} + \frac{bc}{a^2}.$$

Since $K(BNC) = ar_A/2$ and $K(ABC) = (a+b+c)r_A/2$,

$$\frac{K(BNC)}{K(ABC)} = \frac{a}{a+b+c}.$$

We conclude

$$\frac{K(MNP)}{K(ABC)} = \left(1 - \frac{b+c}{a} + \frac{bc}{a^2}\right)\frac{a}{a+b+c} = \frac{1-(b+c)/a + bc/a^2}{1+(b+c)/a}.$$

Let $t = (b+c)/a$, so that $(t^2-1)/2 = bc/a^2$. We express $K(MNP)/K(ABC)$ as $(t-1)^2/(2t+2)$, which we denote by $f(t)$. Since

$$t = \sin C + \cos C = \sqrt{2}\cos(C - \pi/4)$$

and $0 < C < \pi/2$, we have $1 < t \le \sqrt{2}$. Since

$$f'(t) = \frac{(t-1)(t+3)}{2(t+1)^2},$$

we see that f is increasing on $[1, \sqrt{2}]$. Hence f achieves its maximum on $[1, \sqrt{2}]$ at $t = \sqrt{2}$, and that maximum value is

$$f\big(\sqrt{2}\big) = \frac{(\sqrt{2}-1)^2}{2(\sqrt{2}+1)} = \frac{(\sqrt{2}-1)^3}{2},$$

which was to be shown. Note that $t = \sqrt{2}$ when $\cos(C - \pi/4) = 1$, or when $C = \pi/4$, i.e., when the original triangle is isosceles.

Also solved by S. Amghibech (Canada), M. Bataille (France), H. Chen, P. P. Dályay (Hungary), P. De (India), R. Downes, A. Fanchini (Italy), G. Fera (Italy), K. Gatesman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, W. Janous (Austria), B. Karaivanov (USA) & T. S. Vassilev (Canada), K. T. L. Koo (China), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Lukarevski (Macedonia), D. Ş. Marinescu & M. Monea (Romania), C. Mindrila, R. Nandan, C. Pranesachar (India), A. Stadler (Switzerland), R. Stong, K. Sullivan, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

Nontrivial Solutions To a Matrix Equation

12087 [2019, 82]. *Proposed by M. L. J. Hautus, Heeze, Netherlands.* Let K be a field, and let A be a linear map from K^n into itself. The equation $X^2 = AX$ has the trivial solutions $X = 0$ and $X = A$. Show that it has a nontrivial solution if and only if the characteristic polynomial $\det(\lambda I - A)$ is reducible, with the following sole exception: If K has two elements, $n = 2$, and A is nilpotent and nonzero, then the characteristic polynomial is reducible, yet $X^2 = AX$ has no nontrivial solutions.

Solution by Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong, China. The problem is trivial if $n = 1$ or $A = 0$. Suppose $n \ge 2$ and $A \ne 0$.

Suppose that $X^2 = AX$ has a solution X outside $\{0, A\}$, and consider the characteristic polynomial $\det(\lambda I - A)$. Let $V = \{Xv \colon v \in K^n\}$. Note that $V \ne \{0\}$. If $V = K^n$, then X is surjective and hence invertible, so $X = A$. Thus V is a proper subspace of K^n. Since $A(Xv) = (AX)v = X^2v = X(Xv) \in V$ for all $v \in K^n$, we see that A maps V into V. Choose a basis of V and extend it to a basis of K^n. With respect to this basis, A has a matrix representation of the form $\left[\begin{smallmatrix} B & C \\ 0 & D \end{smallmatrix}\right]$, where both B and D are square with order less than n. Thus $\det(\lambda I - A) = \det(\lambda I - B)\det(\lambda I - D)$, so $\det(\lambda I - A)$ is reducible.

For the converse, suppose $\det(\lambda I - A) = p(\lambda)q(\lambda)$, where $\deg(p) = m$ with $1 \le m < n$. By the Cayley–Hamilton theorem, $p(A)q(A) = q(A)p(A) = 0$.

We claim that there is an A-invariant proper subspace of K^n. If $p(A) = 0$ (or similarly if $q(A) = 0$), take a nonzero $w \in K^n$. The span of $\{A^j w : 0 \le j \le m-1\}$ is A-invariant, and the dimension of the span is at most m, which is less than n. On the other hand, if $p(A) \ne 0$ and $q(A) \ne 0$, then $p(A)$ cannot be surjective, because $q(A)p(A) = 0$ and $q(A) \ne 0$. Thus the image of $p(A)$ is proper and A-invariant.

Let W be an A-invariant proper subspace of K^n with dimension k, where $1 \le k < n$. Choose a basis of W and extend it to a basis for K^n. The matrix representation of A with respect to this basis has the form $\begin{bmatrix} P & Q \\ 0 & R \end{bmatrix}$, where P has order k and R has order $n-k$. Setting $X = \begin{bmatrix} P & T \\ 0 & 0 \end{bmatrix}$, we have $X^2 = \begin{bmatrix} P^2 & PT \\ 0 & 0 \end{bmatrix} = AX$.

It remains to choose T so that $X \notin \{0, A\}$. If $n > 2$ or $|K| > 2$, then we can choose $T \notin \{0, Q\}$, giving $X^2 = AX$ nontrivial solutions. When $n = 2$ and $K = \{0, 1\}$, if A is not nilpotent, then $A = \begin{bmatrix} 1 & Q \\ 0 & R \end{bmatrix}$ (with respect to some basis). Let $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ if $Q \ne 0$, and $X = \begin{bmatrix} 1 & T \\ 0 & 0 \end{bmatrix}$ with $T \ne 0$ if $Q = 0$. Again $X^2 = AX$ has solutions outside $\{0, A\}$.

When $n = 2$ and $K = \{0, 1\}$, and A is nilpotent, we have $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (up to similarity). Note that $\det(\lambda I - A) = \lambda^2$ is reducible. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \{0, 1\}$, be a solution to $X^2 = AX$. Computation yields $a = c = d = 0$. Now $X = 0$ if $b = 0$ and $X = A$ if $b = 1$, so $X^2 = AX$ has no nontrivial solutions.

Also solved by D. Fleischman, S. M. Gagola, Jr., O. P. Lossers (Netherlands), J. H. Smith, and the proposer.

An Integral Inequality

12088 [2019, 83]. *Proposed by Florin Stanescu, Serban Cioculescu School, Gaesti, Romania.* Let k be a positive integer with $k \ge 2$, and let $f : [0, 1] \to \mathbb{R}$ be a function with continuous kth derivative. Suppose $f^{(k)}(x) \ge 0$ for all $x \in [0, 1]$, and suppose $f^{(i)}(0) = 0$ for all $i \in \{0, 1, \dots, k-2\}$. Prove

$$\int_0^1 x^{k-1} f(1-x)\, dx \le \frac{(k-1)!\, k!}{(2k-1)!} \int_0^1 f(x)\, dx.$$

Solution by Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong, China. Let $Q(x) = x^{2k-1} - x^k$. We have $Q^{(i)}(0) = 0$ when $0 \le i \le k-1$ and $Q(1) = 0$. Integration by parts k times yields

$$\int_0^1 Q^{(k)}(x) f(1-x)\, dx = \int_0^1 Q(x) f^{(k)}(1-x)\, dx \le 0,$$

since $Q(x) \le 0$ and $f^{(k)}(1-x) \ge 0$ both hold on $[0, 1]$. Substituting

$$Q^{(k)}(x) = (2k-1)(2k-2)\cdots k \cdot x^{k-1} - k!$$

into this gives

$$\frac{(2k-1)!}{(k-1)!} \int_0^1 x^{k-1} f(1-x)\, dx - k! \int_0^1 f(1-x)\, dx \le 0.$$

Therefore

$$\int_0^1 x^{k-1} f(1-x)\, dx \le \frac{(k-1)!\, k!}{(2k-1)!} \int_0^1 f(1-x)\, dx = \frac{(k-1)!\, k!}{(2k-1)!} \int_0^1 f(x)\, dx,$$

where we have used the change of variable $x \mapsto 1-x$ in the last step.

Also solved by U. Abel (Germany), S. Amghibech (Canada), K. F. Andersen (Canada), P. Bracken, B. Bradie, R. Chapman (UK), P. P. Dályay (Hungary), K. Gatesman, E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), D. Ş. Marinescu & M. Monea (Romania), V. Mikayelyan (Armenia), M. Omarjee (France), H. Park, A. Pathak, K. Schilling, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, T. Wiandt, and the proposer.

Ring Homomorphisms

12089 [2019, 83]. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO, and Adam Salminen, University of Evansville, Evansville, IN.* All rings in this problem are assumed to be commutative with a nonzero multiplicative identity. A *homomorphism* from a ring R to a ring S is an identity-preserving map $\phi : R \to S$ such that $\phi(x+y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in R$. Consider the following two properties of a ring R:

(1) For every proper ideal I of R, there is an injective homomorphism $\phi : R/I \to R$.
(2) For every proper ideal I of R, there is an injective homomorphism $\phi : R \to R/I$.

(a) Must a ring that enjoys property (1) be a field?
(b) Must a ring that enjoys property (2) be a field?
(c) Must a ring that enjoys properties (1) and (2) be a field?

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands.
(a) The answer is no. For example, let K be a field and take $R = K[x] \bmod x^2$. The only nontrivial ideal I is (x), and $R/(x) = K$, which embeds into R. Yet R is not a field. Another example is $R = K \oplus K$.

(b) The answer is no again. Let K be a field, and let $\langle x \rangle$ be a sequence such that x_n is transcendental over $K(x_1, \ldots, x_{n-1})$ for $n \geq 1$. Write $F = K(x_2, x_3, \ldots)$ and $E = K(x_1, x_2, \ldots)$, and let R be $F[x_1]$, the polynomial ring in x_1 over F. Any ideal I of R has the form $I = (f(x_1))$, where $f(x_1) = x_1^m - f_{m-1}x_1^{m-1} - \cdots - f_0$. Hence

$$R/I = \{a_0 + \cdots + a_{m-1}x_1^{m-1} : \ a_0, \ldots, a_{m-1} \in F \text{ and } x^m = f_{m-1}x_1^{m-1} + \cdots + f_0\}.$$

Since R embeds into E, E is isomorphic to F, and F embeds into R/I, we see that R embeds into R/I. Yet R is not a field.

(c) Here the answer is yes. First let I be a maximal ideal of R. (Note that I exists by Zorn's lemma.) Now let $K = R/I$; note that K is a field. By (2), R embeds into K. So we may assume $R \subseteq K$ and have $0 = 0_R$ and $1 = 1_R$ in K. Hence R has no zero divisors. Now for $r \in R - \{0\}$, let r^{-1} be the inverse of r in K, and take $I = (r^2)$. We have $r^2 \equiv 0 \bmod I$. By (1), R/I has no zero divisors, so we must have $r \equiv 0 \bmod I$. Hence $r \in I$, so $r = r^2 a$ with $a \in R$. Now $ra = 1$ and $a = r^{-1} \in R$.

Editorial comment. The answer to **(c)** assumes the existence of a maximal ideal, which depends on Zorn's lemma or some other statement equivalent to the axiom of choice.

Also solved by A. J. Bevelacqua, S. Dey, G. Marks, M. Reid, Missouri State University Problem Solving Group, and the proposer.

A Pell–Lucas Computation of Pi

12090 [2019, 180]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* The Pell–Lucas numbers Q_n satisfy $Q_0 = 2$, $Q_1 = 2$, and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$. Prove

$$\sum_{n=1}^{\infty} \arctan\left(\frac{2}{Q_n}\right) \arctan\left(\frac{2}{Q_{n+1}}\right) = \frac{\pi^2}{32}.$$

Solution by M. Vowe, Therwil, Switzerland. The solutions of the equation $x^2 - 2x - 1 = 0$ are $p = 1 + \sqrt{2}$ and $q = 1 - \sqrt{2}$, yielding $Q_n = Ap^n + Bq^n$. Since $Q_0 = Q_1 = 2$, we obtain $Q_n = p^n + q^n$. For simplicity, we write $A(x)$ for $\arctan(x)$. Note the identities $A(u) + A(1/u) = \pi/2$ and

$$A(u) - A(v) = A\left(\frac{u - v}{1 + uv}\right)$$

for $u, v > 0$. When n is odd, $q^n = -p^{-n}$, and so

$$\begin{aligned} A\left(\frac{2}{Q_n}\right) &= \frac{\pi}{2} - A\left(\frac{Q_n}{2}\right) = \frac{\pi}{2} - A\left(\frac{p^n - p^{-n}}{2}\right) \\ &= \frac{\pi}{2} - \left(A(p^n) - A(p^{-n})\right) = \frac{\pi}{2} - \left(2A(p^n) - \frac{\pi}{2}\right) = \pi - 2A(p^n). \end{aligned}$$

When n is even, $q^n = p^{-n}$, and so

$$\begin{aligned} A\left(\frac{2}{Q_n}\right) &= A\left(\frac{2}{p^n + q^n}\right) = A\left(\frac{2p^n}{1 + p^{2n}}\right) \\ &= A\left(\frac{p^n(p - 1/p)}{1 + p^{2n}}\right) = A\left(\frac{p^{n+1} - p^{n-1}}{1 + p^{2n}}\right) = A(p^{n+1}) - A(p^{n-1}). \end{aligned}$$

Now the even partial sums of the specified series are given by

$$\begin{aligned} S_{2N} &= \sum_{k=1}^{N} \left(\pi - 2A\left(p^{2k-1}\right)\right)\left(A\left(p^{2k+1}\right) - A\left(p^{2k-1}\right)\right) \\ &\qquad + \sum_{k=1}^{N} \left(A\left(p^{2k+1}\right) - A\left(p^{2k-1}\right)\right)\left(\pi - 2A\left(p^{2k+1}\right)\right) \\ &= 2\pi \sum_{k=1}^{N} \left(A\left(p^{2k+1}\right) - A\left(p^{2k-1}\right)\right) - 2\sum_{k=1}^{N} \left(A^2\left(p^{2k+1}\right) - A^2\left(p^{2k-1}\right)\right) \\ &= 2\pi\left(-A(p) + A\left(p^{2N+1}\right)\right) - 2\left(-A^2(p) + A^2\left(p^{2N+1}\right)\right). \end{aligned}$$

Since

$$-1 = \tan(2(3\pi/8)) = \frac{2\tan(3\pi/8)}{1 - \tan^2(3\pi/8)},$$

$A(p) = A(1 + \sqrt{2}) = 3\pi/8$. Finally, since $\lim_{n\to\infty} A(p^n) = \pi/2$,

$$\lim_{N\to\infty} S_N = 2\pi\left(-\frac{3\pi}{8} + \frac{\pi}{2}\right) - 2\left(-\frac{9\pi^2}{64} + \frac{\pi^2}{4}\right) = \frac{\pi^2}{32}.$$

Editorial comment. Giuseppe Fera obtained a similar result for Fibonacci numbers, with $2/Q_n$ replaced by $1/F_n$, yielding the sum $\pi^2/8$.

Also solved by P. Bracken, B. Bradie, P. Budney, R. Chapman (UK), G. Fera (Italy), D. Garth, K. Gatesman, K. T. L. Koo (China), O. P. Lossers (Netherlands), R. Molinari, M. Omarjee (France), M. A. Prasad (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, Y. Xiang (China), J. Zacharias, L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.