Chapter 4

Principle of Mathematical Induction

Theorem 4.1. *(Principle of Mathematical Induction) Let S_n be a statement about a positive integer n. Suppose that*

1. *S_1 is true,*
2. *If $k \geq 1$ and S_k is true, then S_{k+1} is true.*

Then S_n is true for all positive integers n.

Note. Conditions 1 and 2 in the above theorem are called the basis step and inductive step respectively.

This principle is easy to understand using the following example: suppose we know how to get to the first floor of a building (e.g. we know where an entrance is), and we also know how to get from any floor to the next one (e.g. we know where an elevator or a staircase is). Then we'll be able to get to any floor in this building. Namely, we'll be able get to the first floor, and then from the first to the second, and then from the second to the third, and so on. The same is true for any statement. If we can check that S_1 is true, then the second condition in theorem 4.1 ensures that S_2 follows from S_1, S_3 follows from S_2, and so on. Thus S_n is true for any natural number n.

Mathematical Induction is used in all areas of mathematics. It can be used to prove summation formulas such as in the next example, various number theory, algebraic, and geometric statements.

Example 4.2. Prove that for any natural number n,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Proof. We will prove this identity using Mathematical Induction.
Basis step: if $n = 1$, the formula says that $1 = \dfrac{1 \cdot (1+1)}{2}$ which is true.
Inductive step: suppose the formula holds for $n = k$, i.e. that

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2} \tag{4.1}$$

is true. We have to show that the formula holds for $n = k + 1$, i.e. that

$$1 + 2 + 3 + \ldots + (k + 1) = \frac{(k + 1)((k + 1) + 1)}{2}$$

is true. Adding $k + 1$ to both sides of (4.1) gives:

$$\begin{aligned} 1 + 2 + 3 + \ldots + k + (k + 1) &= \frac{k(k + 1)}{2} + (k + 1) \\ &= \frac{k(k + 1) + 2(k + 1)}{2} \\ &= \frac{(k + 2)(k + 1)}{2} \\ &= \frac{((k + 1) + 1)(k + 1)}{2}. \end{aligned}$$

□

Note. For any specific value of n, it is easy to check that the identity holds. For example, for the first four natural numbers we have:

$$1 = \frac{1 \cdot (1 + 1)}{2}, \quad 1 + 2 = \frac{2 \cdot (2 + 1)}{2}, \quad 1 + 2 + 3 = \frac{3 \cdot (3 + 1)}{2}, \quad 1 + 2 + 3 + 4 = \frac{4 \cdot (4 + 1)}{2}.$$

However, remember that it is not sufficient to check *some* values of n. We had to prove the statement for *all* natural numbers n.

Remark. We might want to prove a statement S_n for all $n \geq 0$, or for all $n \geq 2$, etc., rather than for all $n \geq 1$. In this case, the basis step should check that the statement is valid for the smallest value of n, say, $n = 0$, or $n = 2$ in the above cases, and the inequality $k \geq 1$ in the inductive step should be modified accordingly ($k \geq 0$, or $k \geq 2$, etc.).

Sometimes to prove S_{k+1}, it is insufficient to assume S_k alone, but S_n for $n \leq k$ is needed. Then we use the so-called Strong Induction formulated below.

Theorem 4.3. *(Strong Mathematical Induction) Let S_n be a statement about a positive integer n. Suppose that*

1. *S_1 is true,*
2. *If $k \geq 1$ and S_n is true for all $1 \leq n \leq k$, then S_{k+1} is true.*

Then S_n is true for all positive integers n.

Remark. As above, we might want to start with 0 or 2 or something else rather than with 1.

Example 4.4. Prove that any integer $n \geq 2$ can be written in the form $n = 2a + 3b$ for some nonnegative integers a and b (we will say that n is a nonnegative linear combination of 2 and 3).

Proof. Basis step. If $n = 2$, we have $n = 2 \cdot 1 + 3 \cdot 0$.
Inductive step. Suppose that $k \geq 2$ and the statement holds for all $2 \leq n \leq k$. We want to prove it for $n = k + 1$.
Case I. $k = 2$, so $k + 1 = 3$. Then $k + 1 = 3 = 2 \cdot 0 + 3 \cdot 1$.
Case II. $k \geq 3$, then $2 \leq k - 1 \leq k$, thus the statement holds for $n = k - 1$. We have $k - 1 = 2a + 3b$ for some nonnegative integers a and b. Then $k + 1 = k - 1 + 2 = 2a + 3b + 2 = 2(a + 1) + 3b$, so $k + 1$ is a nonnegative linear combination of 2 and 3. □

Remark. Notice that case I above simply checks that the statement holds for $n = 3$. In literature, this calculation is often moved to the basis step.

Problems

1. Prove that the following formulas hold for any natural n.

   (a) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

   (b) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

   (c) $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$

   (d) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

   (e) $1 + 3 + 5 + \ldots + (2n-1) = n^2$

2. Prove that for any positive integer n, $n < 2^n$.

3. Prove that if q is a positive integer, then $3^{2^q} - 1$ is divisible by 2^{q+2}.

4. Suppose that $2n$ points are given in space, where $n \geq 2$. Altogether $n^2 + 1$ line segments are drawn between these points. Prove that there is at least one triangle (a set of three points which are joined pairwise by line segments).

5. Let $\{F_0,\ F_1,\ F_2,\ \ldots\}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$. Prove the following identities.

   (a) $F_1F_2 + F_2F_3 + \ldots + F_{2n-1}F_{2n} = F_{2n}^2$

   (b) $F_1^2 + F_2^2 + \ldots + F_n^2 = F_nF_{n+1}$

   (c) $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$

   (d) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$

   (e) $F_{n-1}^2 + F_n^2 = F_{2n-1}$

6. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from other cars on its way around.

7. Every road in Sikinia is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly or via at most one other city.

8. Suppose that n lines are given in the plane. They divide the plane into regions. Show that it is possible to color the plane with two colors so that no regions with a common boundary line are colored the same way. Such a coloring is called a proper coloring.

9. Consider a few points in the plane and a few line segments connecting some of them so that (1) no two line segments intersect, and (2) each point is connected with at least two other points (so there are no isolated points and there are no "hanging" line segments). Such line segments divide the plane into several regions. Such a picture is called a map. Prove that a map can be properly colored with two colors if and only if each point is connected with an even number of other points. (See problem 8 for definition of a proper coloring)

10. Let α be any real number such that $\alpha + \dfrac{1}{\alpha} \in \mathbb{Z}$. Prove that $\alpha^n + \dfrac{1}{\alpha^n} \in \mathbb{Z}$ for any $n \in \mathbb{N}$.

11. Prove that $1 < \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{3n+1} < 2$.

12. Let n be any natural number. Consider all nonempty subsets of the set $\{1, 2, \ldots, n\}$, which do not contain any neighboring elements. Prove that the sum of the squares of the products of all numbers in these subsets is $(n+1)! - 1$. (For example, if $n = 3$, then such subsets of $\{1, 2, 3\}$ are $\{1\}$, $\{2\}$, $\{3\}$, and $\{1, 3\}$, and $1^2 + 2^2 + 3^2 + (1 \cdot 3)^2 = 23 = 4! - 1$.)

13. Prove that the determinant of the $n \times n$ matrix M_n with entries

$$m_{ij} = \begin{cases} 5 \text{ if } i = j \\ 2 \text{ if } |i - j| = 1 \\ 0 \text{ otherwise} \end{cases}$$

is equal to $\dfrac{1}{3}(4^{n+1} - 1)$.

14. Find the determinant of the $n \times n$ matrix A_n with entries

$$a_{ij} = \begin{cases} 2 \text{ if } i = j \\ 1 \text{ if } |i - j| = 1 \\ 0 \text{ otherwise} \end{cases}.$$

Hint: calculate the determinants of A_1, A_2, A_3, and A_4. Notice the pattern. Guess a formula for $\det A_n$, and then prove it by Mathematical Induction.

15. Prove that if any one square of a $2^n \times 2^n$ chessboard is removed, then the remaining board can be covered by L-trominoes, i.e. the figures consisting of 3 squares as shown below.