STRATEGIES OF PROBLEM SOLVING

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Chapter 1

Introduction

Solving mathematical problems is both a science and an art. It is a science because we need to learn some basic concepts and skills, and use proper terminology when explaining our solution to other people. It is also an art because very often we need to be creative. There are infinitely many types of math problems. While it is important to learn some basic principles of problem solving, it is impossible to learn how to solve every problem in the world. Just like it is impossible to learn how to construct every possible model using Lego Blocks. However, after you've constructed a few following the directions in your Lego set, you can create your own. When you finish school and move on to solving real world problems, you may need some initial training in your field, but no training can ever give you detailed instructions on what to do in every possible situation. You'll have to think, make decisions, try things out, and learn from your successes and mistakes. This is what we are going to do in this book: learn some basics and then explore on our own and learn from our experience.

Below are some problems. Let's try solving them. Solutions to these problems will illustrate some things mentioned above as well as will introduce a few concepts and principles that will be discussed in later chapters.

1. Eleven children contributed money to buy a present for their classmate. The total amount of money collected was \$30.00. Prove that at least one child gave at least \$2.73.

2. (a) Prove that any two-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

   (b) Prove that any natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.

3. Is it true or false that for any natural number n, the number $n^2 + n + 41$ is prime?

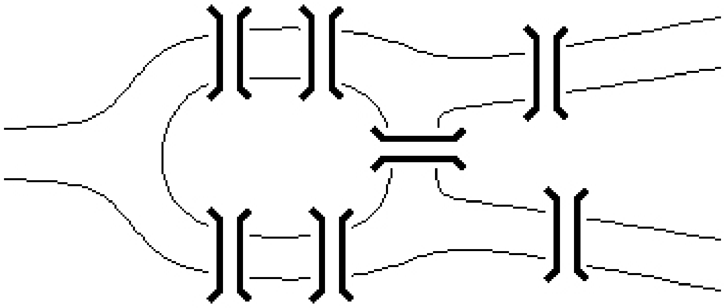
4. In a 4×4 table six cells are marked by a star and all others are blank. Show that it is possible to cross out 2 columns and 2 rows so that the remaining cells are blank.

5. Is it true or false that for any natural number n, the number $n^3 + 2n$ is divisible by 3?

6. Sketch the graph of $f(x) = |x + 2| + |2x - 5|$.

7. Konigsberg is a city which was the capital of East Prussia but now is known as Kaliningrad in Russia. The city is built around the River Pregel where it joins another river. An island named Kniephof is in the middle of where the two rivers join. There are

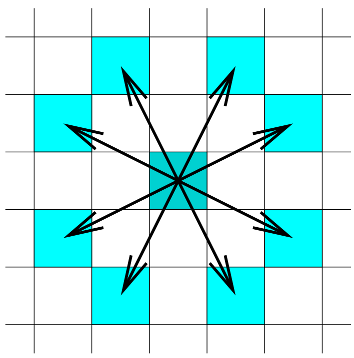
seven bridges that join the different parts of the city on both sides of the rivers and the island.



People tried to find a way to walk all seven bridges without crossing a bridge twice, but no one could find a way to do it. The problem came to the attention of a Swiss mathematician named Leonhard Euler. In 1735, Euler presented the solution to the problem before the Russian Academy.

Now, you too try to solve this problem. If such a tour exists, find it. If not, explain why not.

8. (a) Is it possible for a chess knight to start at the upper left corner and go through every square on the 8×8 chessboard exactly once? (A knight's move is 2 squares up, down, or to the right or left, and 1 square in a perpendicular direction. All allowed moves from a certain square are shown below.)



   (b) Is it possible for a knight to start at the upper left corner, go through every square on the 8×8 chessboard exactly once, and come back to the starting point?

As said above, learning to solve problems is in part difficult because problems can be very different. However, there are a few basic principles that are good to know. There are a few approaches and methods that can be useful. In this book, we'll study some of them. After you study the material of this book you should be able to solve many problems pretty easily.

While using intuition and working out a few examples may help us find an idea, it is also important to write rigorous proofs. Since our intuition is not always correct, we need to justify each step in a solution. We will therefore try to avoid words such as 'obviously'.

In each chapter, we provide basic definitions and facts to get you started. We do not prove most of the facts given in this book, since our main goal is to learn how to solve problems, i.e. use these facts. You will probably prove most, if not all, the facts given in this book in courses such as Calculus, Discrete Mathematics, Abstract Algebra, and Number Theory. Sometimes the idea of a proof of a theorem can be used for solving many problems. In such cases we provide the proof.

Chapter 2

Introduction to Logic

In this chapter we will introduce basic logic terminology and notations. It will be useful when we discuss types of proofs (see chapter **??**).

Definition 2.1. A **proposition** is a declarative sentence that is either true or false.

For example, "3 plus 2 is 5" is a true proposition, "3 times 2 is 7" is a false proposition, while "x minus 4 is 8" is not a proposition because the value of x has not been defined, and "is 3 plus 3 equal 6?" is not a proposition because it is an interrogative, not declarative, sentence.

Definition 2.2. Let p and q be propositions. Then:

- The **negation** of p, denoted by $\neg p$, is the proposition "not p". It is true if and only if p is false.
- The **conjunction** of p and q, denoted by $p \wedge q$, is the proposition "p and q". It is true if and only if both p and q are true.
- The **disjunction** of p and q, denoted by $p \vee q$, is the proposition "p or q". It is true if and only if at least one of p and q is true. Note that if both p and q are true, then $p \vee q$ is true, so "or" is not exclusive.
- The **exclusive or** of p and q, denoted by $p \oplus q$, is the proposition "either p or q but not both". It is true if and only if exactly one of p and q is true.
- The **implication** of p and q, denoted by $p \to q$, is the proposition "if p then q". It is false when p is true and q is false, and true otherwise.
- The **biconditional** of p and q, denoted by $p \leftrightarrow q$, is the proposition "p if and only if q". It is true when p and q have the same truth values and is false otherwise.

Below is the so-called truth table that shows the truth values of the compound propositions defined above depending on the truth values of p and q.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Notice that $p \to q$ is false if and only if p is true and q is false. We will need this observation in chapter **??**.

- Symbols $\neg$, $\wedge$, $\vee$, $\oplus$, $\rightarrow$, and $\leftrightarrow$ are called **logical connectives**.
- A compound proposition that is always true, no matter what the truth values of the propositions that occur in it are, is called a **tautology**.

  For example, $p \vee \neg p$ is a tautology.
- A compound proposition that is always false is called a **contradiction**.

  For example, $p \wedge \neg p$ is a contradiction.
- Propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \Leftrightarrow q$ denotes that p and q are logically equivalent. Note that p and q are logically equivalent if and only if they always have the same truth values.

Example 2.3. Show that $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$ are logically equivalent.

Solution. *We construct the truth table:*

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

We see that the truth values of $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$ are always the same, therefore these propositions are logically equivalent.

Definition 2.4. A statement $P(x)$ that depends on the value of a variable (x in this case) is called a **propositional function**. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

For example, if $P(x)$ is the statement "$x > 3$", then $P(4)$ is true and $P(2)$ is false.

Definition 2.5. Let $P(x)$ be a propositional function. Then

- $\forall x P(x)$ means "for every x, $P(x)$ is true".
- $\exists x P(x)$ means "there exists a value of x for which $P(x)$ is true".
- $\exists! x P(x)$ means "there exists a unique value of x for which $P(x)$ is true".

Symbols $\forall$ and $\exists$ are called **quantifiers**. Namely, $\forall$ is called a **universal quantifier**, and $\exists$ is called an **existential quantifier**.

When interpreting expressions $\forall x P(x)$, $\exists x P(x)$, $\exists! x P(x)$, we need to specify a set S of all possible choices of x. Such a set is called the **domain of discourse**. Unless the domain of discourse has already been specified or is clear from context, we can write $\forall x \in S\ P(x)$, etc. to make it explicit. For example, "the square of every integer x is nonnegative" can be written as $\forall x \in \mathbb{Z}\ x^2 \geq 0$.

Propositional functions can be functions of two or more variables, and then we can use two or more quantifiers with them. It is important to realize that the order of quantifiers makes a difference. For example, below we will use the propositional function $F(x, y)$ which means that x and y are friends (the domain of this function can be a set of people). Then e.g. $\forall x \exists y F(x, y)$ means that everybody has at least one friend, while $\exists y \forall x F(x, y)$ means that there is a person who is friends with everybody.

Propositions with negations can always be written so that no negation is outside a quantifier or an expression involving logical connectives, for example:

- $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
- $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
- $\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$
- $\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$

Problems

1. Show that the following propositions are logically equivalent.
   (a) $p \rightarrow q$ and $\neg q \rightarrow \neg p$
   (b) $p \rightarrow q$ and $\neg p \vee q$
   (c) $\neg(p \wedge q)$ and $\neg p \vee \neg q$
   (d) $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$
2. Which of the following sentences are statements? For those that are, indicate the truth value.
   (a) Five plus eight is thirteen.
   (b) Five minus eight is three.
   (c) Two times x is 6.
   (d) The number 2n+6 is an even integer.
   (e) There are 200 elephants in the San Diego Wild Animal Park.
   (f) I have solved all problems in chapter 1.
   (g) Did you do your homework today?
3. Translate the statement

   $$\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$$

   into English, where $C(x)$ is "x has a computer", $F(x, y)$ is "x and y are friends", and the domain of discourse is the set of all students at your university.
4. Let $F(x, y)$ be statement "x can fool y". Use quantifiers to express each of the following statements:
   (a) Everybody can fool Amy.
   (b) Mike can fool everybody.
   (c) Everybody can fool somebody.
   (d) There is no one who can fool everybody.
   (e) Everyone can be fooled by somebody.
   (f) No one can fool both Kate and Jerry.
   (g) Tim can fool exactly two people.
   (h) There is exactly one person whom everybody can fool.
   (i) No one can fool himself or herself.

5. Let $P(x)$ denote the propositional function "$x = -5$" and let $Q(x)$ denote the propositional function "$x^2 = 25''$ and let the domain of discourse be the set of real numbers. Determine the truth values of the following propositions:

   (a) $P(4)$

   (b) $P(4) \to Q(4)$

   (c) $\exists x \neg P(x)$

   (d) $\forall x(P(x) \lor Q(x))$

   (e) $\exists x(P(x) \land Q(x))$

   (f) $\forall x(P(x) \to Q(x))$

   (g) $\exists x(P(x) \to Q(x))$

   (h) $\forall x(P(x) \leftrightarrow Q(x))$

6. Let $P(x)$ denote the propositional function "$(x < 3) \lor (x > 5)$" and let the domain of discourse be the set of real numbers. Determine the truth values of the following propositions.

   (a) $P(2)$

   (b) $P(4)$

   (c) $P(2) \land P(4)$

   (d) $\forall x P(x)$

   (e) $\exists x P(x)$

   (f) $\exists! x P(x)$

   (g) $\forall x(P(x) \lor P(-x))$

7. Let $Q(x, y)$ denote "$x + y = 0$" and let the domain of discourse be the set of real numbers. What are the truth values of the statements $\forall x \exists y Q(x, y)$ and $\exists y \forall x Q(x, y)$?

8. Rewrite each of the following statements so that negations appear only immediately before propositional functions.

   (a) $\neg \forall x \forall y P(x, y)$

   (b) $\neg \forall y \exists x P(x, y)$

   (c) $\neg \forall y \forall x(P(x, y) \lor Q(x, y))$

   (d) $\neg(\exists x \exists y \neg P(x, y) \land \forall x \forall y Q(x, y))$

   (e) $\neg \forall x(\exists y \forall z P(x, y, z) \land \exists z \forall y P(x, y, z))$

   (f) $\neg \exists! x P(x)$

9. Let $P(x, y)$ denote the proposition "$x < y$" and let the domain of discourse be the set of real numbers. Determine the truth values of the following propositions.

   (a) $\exists x \exists y P(x, y)$,

   (b) $\forall x \exists y P(x, y)$,

   (c) $\exists x \forall y P(x, y)$,

   (d) $\forall x \forall y P(x, y)$,

   (e) $\forall x P(-x, x)$.

10. Let $R(x,y)$ be the statement "$x + y = x - y$" and let the domain of discourse be the set of integers. Find the truth values of the following statements. Explain.
    (a) $R(2,0)$
    (b) $\forall y R(1,y)$
    (c) $\forall x \exists y R(x,y)$
    (d) $\forall y \exists x R(x,y)$
    (e) $\exists y \forall x R(x,y)$
11. Which of the following compound propositions are logically equivalent, i.e. have the same truth values of any propositional functions $P(x)$ and $Q(x)$? If propositions are logically equivalent, explain why. If not, give an example of propositional functions $P(x)$ and $Q(x)$ for which one of the propositions is true and the other one is false.
    (a) $\forall x(\neg P(x))$ and $\neg(\forall x P(x))$
    (b) $\forall x(P(x) \vee Q(x))$ and $(\forall x P(x)) \vee (\forall x Q(x))$
    (c) $\forall x(P(x) \wedge Q(x))$ and $(\forall x P(x)) \wedge (\forall x Q(x))$
    (d) $\forall x(P(x) \to Q(x))$ and $(\forall x P(x)) \to (\forall x Q(x))$
    (e) $\forall x(P(x) \leftrightarrow Q(x))$ and $(\forall x P(x)) \leftrightarrow (\forall x Q(x))$
    (f) $\exists x(\neg P(x))$ and $\neg(\exists x P(x))$
    (g) $\exists x(P(x) \vee Q(x))$ and $(\exists x P(x)) \vee (\exists x Q(x))$
    (h) $\exists x(P(x) \wedge Q(x))$ and $(\exists x P(x)) \wedge (\exists x Q(x))$
    (i) $\exists x(P(x) \to Q(x))$ and $(\exists x P(x)) \to (\exists x Q(x))$
    (j) $\exists x(P(x) \leftrightarrow Q(x))$ and $(\exists x P(x)) \leftrightarrow (\exists x Q(x))$
12. Express the definition of the limit $\lim_{x \to a} f(x) = L$ using quantifiers.
13. Express the definition of a convergent sequence $a_1, a_2, \ldots$ using quantifiers.
14. In one country there are two cities, A and B, that are only a few miles apart, and whose residents often visit each other. All residents of city A always say the truth, while all residents of city B always lie. A stranger is passing through one of these cities, but he doesn't know which one. How could he, by asking the first man he sees only one question, determine which city he is passing through?

Chapter 3

Types of proofs

In this chapter we summarize basic types of proofs, and then give a few examples to illustrate them.

Suppose we want to prove a proposition p.

- a **direct** proof just shows that p holds;
- a proof **by contradiction** assumes that p is false and derives a contradiction. The contradiction is usually of the form $r \wedge \neg r$ for some proposition r.

If we want to prove an implication "if p, then q", then any of the following types of proofs may be used:

- a **direct** proof just shows how q follows from p;
- a proof **by contradiction** assumes that $p \to q$ is false, i.e. p is true and q is false, and derives a contradiction;
- a proof **by contrapositive** shows that $\neg q$ implies $\neg p$.

A proof of a statement of the form "$\exists x P(x)$" can be

- **constructive** - when we provide (construct) such an x explicitly;
- **existential**, or **nonconstructive** - when we show the existence of such an x without actually constructing it.

To prove a statement of the form "$\forall x P(x)$" where the domain of discourse is a subset of integer numbers, it is often (but not always!) a good idea to use Mathematical Induction (see chapter **??**).

To prove a statement of the form "$p \leftrightarrow q$", we can either

- prove $p \to q$ and $q \to p$ separately, or
- have each step of our proof of the form "if and only if".

To **disprove** a statement means to show that it is false. To disprove a statement of the form $\forall x P(x)$ it is sufficient to show that there exists at least one **counterexample**, that is, there exists at least one case when the statement does not hold.

Below are some examples of various types of proofs listed above.

Example 3.1. Prove that every odd integer is the difference of two perfect squares.

Direct proof: Every odd integer has the form $2n+1$ for some integer n. Observe that $2n+1=(n+1)^2-n^2$.

Example 3.2. Prove that $\sqrt{2}$ is irrational.

Proof by contradiction: Suppose $\sqrt{2}$ is rational. Then there exists an irreducible fraction $\frac{p}{q}=\sqrt{2}$. (Irreducible means that the greatest common divisor of p and q is 1.) Then $\frac{p^2}{q^2}=2$, thus $p^2=2q^2$. If follows that p^2 is even, so p is even. Let $p=2m$, where m is an integer, then $p^2=4m^2$. We have $4m^2=2q^2$, or $2m^2=q^2$. Now we see that q^2 is even, therefore q is even. We get a contradiction because we have that on one hand, p and q have the greatest common divisor 1, but on the other hand p and q are both even.

Example 3.3. Prove that if a and b are integers and ab is even, then either a or b is even (or both).

Proof by contrapositive: Suppose that neither a nor b is even, and we will prove that ab is not even. That is, we suppose that both a and b are odd, and we will prove that ab is odd. Any odd numbers a and b can be written in the form $a=2n+1$ and $b=2m+1$ for some integers n and m. Then we have $ab=(2n+1)(2m+1)=4nm+2n+2m+1=2(2nm+n+m)+1$ is an odd number.

Example 3.4. Prove that for every positive integer n, there exist n consecutive composite numbers.

Constructive proof: We claim that $(n+1)!+2$, $(n+1)!+3$, ... , $(n+1)!+(n+1)$ are all composite. Indeed, $(n+1)!$ is divisible by 2, by 3, ... , and by $n+1$. Therefore $(n+1)!+2$ is divisible by 2, $(n+1)!+3$ is divisible by 3, ... , $(n+1)!+(n+1)$ is divisible by $n+1$.

Example 3.5. Prove that $x^3+x-1=0$ has a real root.

Nonconstructive proof: Let $f(x)=x^3+x-1$. Then $f(-1)=-3<0$ and $f(1)=1>0$. Since $f(x)$ is a polynomial, it is continuous. By the Intermediate Value Theorem, there exists c between -1 and 1 such that $f(c)=0$.

Example 3.6. Prove or disprove that every odd integer is prime.

Counterexample: 9 is odd but not prime. Thus the statement is false.

Problems

1. Prove that if n is an integer and $3n+5$ is odd, then n is even. Is your proof direct, by contradiction, or by contrapositive?

2. Prove that an integer a is even if and only if a^2 is even. Did you prove the two implications separately or simultaneously?

3. Prove or disprove that 2^n+1 is prime for all nonnegative integers n.

4. Prove that for any integer n there is a prime number greater than n. Is your proof constructive?

5. Every odd number is either of the form $4n+1$ (if it has remainder 1 when divided by 4) or of the form $4n+3$ (if it has remainder 3) where n is an integer. Prove that if an odd number is a perfect square, then it has the form $4n+1$. What type of proof did you use? State the converse. Prove or disprove the converse.

6. Prove or disprove that if a and b are rational numbers, then a^b is also rational.
7. Prove that the equation $x^{101} + x^{51} + x + 1 = 0$ has exactly one real solution. Split this into two statements:
   (a) the equation has at least one solution. Is your proof constructive or nonconstructive?
   (b) the equation can not have two distinct roots. Is your proof direct, by contradiction, or by contrapositive?
8. Prove that if the sum of two numbers is irrational then at least one of the numbers is irrational. Is your proof direct, by contradiction, or by contrapositive? State the converse. Prove or disprove the converse.
9. Prove that the equation $4\sin^2 x = 1$ has a real solution. Is your proof constructive?
10. Prove that the equation $x + \sin x = 1$ has a real solution. Is your proof constructive?
11. Prove that the equation $x^2 + x + 1 = 0$ has no rational solutions. Is your proof direct, by contradiction, or by contrapositive?
12. Prove that 0 is a root of the equation $a_n x^n + \ldots a_1 x + a_0 = 0$ if and only if the free term $a_0 = 0$. Did you prove the two implications separately or simultaneously?
13. Prove that if a positive integer is divisible by 8 then it is the difference of two perfect squares. Is your proof direct, by contradiction, or by contrapositive? Is it constructive or nonconstructive?
14. Prove or disprove that if a and b are irrational numbers, then a^b is also irrational.
15. Prove that for any integers n and m, if $nm + 2n + 2m$ is odd then both n and m are odd. Is your proof direct, by contradiction, or by contrapositive?