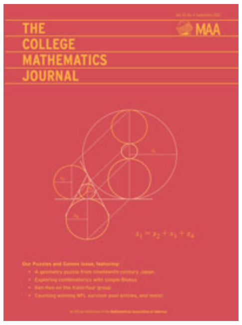

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Problems and Solutions

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PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than March 15, 2023. Sending both pdf and TEX files is ideal.

PROBLEMS

1231. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let ABC be a triangle. Show that $\sum_{\alpha=\angle A,\angle B,\angle C} \sin^3(\alpha)\cos(\alpha) \le \frac{9\sqrt{3}}{16}$.

1232. *Proposed by Jacob Guerra, Salem State University, Salem, MA.*

Define, for every non-negative integer n, the nth Catalan number by $C_n := \frac{1}{n+1}\binom{2n}{n}$. Consider the sequence of complex polynomials in z defined by $z_k := z_{k-1}^2 + z$ for every non-negative integer k, where $z_0 := z$. It is clear that z_k has degree 2^k and thus has the representation $z_k = \sum_{n=1}^{2^k} M_{n,k} z^n$, where each $M_{n,k}$ is a positive integer. Prove that $M_{n,k} = C_{n-1}$ for $1 \le n \le k+1$.

1233. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Suppose that X and Y are independent, uniform random variables over $[0, 1]$. Define U_X, V_X, and B_X as follows: U_X is uniform over $[0, X]$, V_X is uniform over $[X, 1]$, and $B_X \in \{0, 1\}$, with $P(B_X = 1) = X$, and $P(B_X) = 0 = 1 - X$. Now define random variables Z and W_X as follows:

$$Z = (Y - X)\mathbf{1}\{Y \ge X\} + (1 - X + Y)\mathbf{1}\{Y < X\}, \text{ and}$$

$$W_X = B_X \cdot U_X + (1 - B_X)V_X.$$

Prove that both Z and W_X are uniform over $[0, 1]$. Here, $\mathbf{1}[S]$ is the indicator function that is equal to 1 if S is true and 0 otherwise.

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1234. *Proposed by Moubinool Omarjee, Lycée Henry IV, Paris, France.*

For every positive integer n, set $a_n := \sum_{k=1}^{n} \frac{1}{k^4}$ and $b_n := \sum_{k=1}^{n} \frac{1}{(2k-1)^4}$. Compute $\lim_{n\to\infty} n^3(\frac{b_n}{a_n} - \frac{15}{16})$.

1235. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let S be a set, and let $f\colon S \to S$ be a function. For $s \in S$, the *orbit* of s is defined by $\mathcal{O}(s) := \{f^n(s)\colon n \geq 0\}$, where $f^0\colon S \to S$ is the identity map and f^n is the n-fold composition of f with itself for $n > 0$. A subset $X \subseteq S$ is *closed under* f provided that for all $x \in X$, also $f(x) \in X$. Finally, if X is closed under F, we say that X is *finitely generated* if there is a finite $F \subseteq X$ such that $X = \bigcup_{x\in F} \mathcal{O}(x)$. Find all structures (S, f) up to isomorphism where S is not finitely generated, but every proper subset of S closed under f is finitely generated. Note that (S, f) and (T, g) are *isomorphic* if there is a bijection $\varphi\colon S \to T$ such that $\varphi(f(s)) = g(\varphi(s))$ for all $s \in S$.

SOLUTIONS

Harmonic, Fibonacci, and triangular numbers

1206. *Proposed by Seán M. Stewart, Bomaderry, NSW, Australia.*

Let $H_n := \sum_{k=1}^{n} \frac{1}{k}$ denote the nth harmonic number, let F_n denote the nth Fibonacci number, where $F_0 := 0$, $F_1 := 1$, and $F_n := F_{n-1} + F_{n-2}$ for $n \geq 2$. Further, let T_n be the nth triangular number defined by $T_0 := 0$ and $T_n := n + T_{n-1}$ for $n \geq 1$, and let $\varphi := \frac{1+\sqrt{5}}{2}$ be the golden ratio. Prove the following:

$$\sum_{n=1}^{\infty} \frac{T_n H_n F_n}{2^n} = 52\log(2) + \frac{232}{\sqrt{5}}\log(\varphi) + 73.$$

Solution by Hongwei Chen, Christopher Newport University, Newport News, Virginia.

Recall the generating function of the harmonic numbers:

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\log(1-x)}{1-x}.$$

Let

$$f(x) := -\frac{x\log(1-x)}{1-x}.$$

Differentiating

$$\sum_{n=1}^{\infty} H_n x^{n+1} = f(x)$$

twice leads to

$$\sum_{n=1}^{\infty} n(n+1)H_n x^{n-1} = f''(x). \tag{1}$$

Using this fact and $T_n = n(n+1)/2$ we find the generating function of $\{T_n H_n\}_{n=1}^{\infty}$:

$$\sum_{n=1}^{\infty} T_n H_n x^n = \frac{1}{2} x f''(x) := g(x).$$

Direct computation gives

$$g(x) = \frac{x}{2}\left(\frac{2}{(1-x)^2} + \frac{3x}{(1-x)^3} - \frac{2\log(1-x)}{(1-x)^2} - \frac{2x\log(1-x)}{(1-x)^3}\right).$$

Using the well-known Binet formula

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \left(-\frac{1}{\phi}\right)^n\right),$$

and with some simplifications, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{T_n H_n F_n}{2^n} &= \frac{1}{\sqrt{5}}\left(g\left(\frac{\phi}{2}\right) - g\left(-\frac{1}{2\phi}\right)\right)\\ &= 73 - \frac{130 - 58\sqrt{5}}{5}\log\left(1 + \frac{1}{2\phi}\right) - \frac{130 + 58\sqrt{5}}{5}\log\left(1 - \frac{\phi}{2}\right) \quad (1)\end{aligned}$$

Notice that

$$\log\left(1 + \frac{1}{2\phi}\right) + \log\left(1 - \frac{\phi}{2}\right) = \log\left(1 + \frac{1}{2\phi}\right)\left(1 - \frac{\phi}{2}\right) = \log\left(\frac{1}{4}\right) = -2\log(2)$$

and

$$\log\left(1 + \frac{1}{2\phi}\right) - \log\left(1 - \frac{\phi}{2}\right) = \log\left(\frac{1 + 1/2\phi}{1 - \phi/2}\right) = \log(\phi^4) = 4\log(\phi).$$

From (1) we consequently find

$$\sum_{n=1}^{\infty} \frac{T_n H_n F_n}{2^n} = 73 + 52\log(2) + \frac{232}{\sqrt{5}}\log(\phi),$$

as desired.

Also solved by Narendra Bhandari, Bajura, Nepal; Brian Bradie,Christopher Newport U.; Bruce Burdick, Providence, RI; Nandan Sai Dasireddy, Hyderabad, Telangana, India; Russ Gordon, Whitman C.; Eugene Herman, Grinnell C.; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Volkhard Schindler, Berlin, Germany; Albert Stadler, Herrliberg, Switzerland; Enrique Treviño, Lake Forest C.; and the proposer.

A sum of a product of sums

1207. *Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Establish the following:

$$\sum_{n=1}^{\infty}(2n-1)\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)=\zeta(2)+\zeta(3),$$

where for a positive integer k, we have $\zeta(k)=\sum_{n=1}^{\infty}\frac{1}{n^k}$.

Solution by Brian Bradie, Christopher Newport University, Newport News, Virginia.

First, write

$$\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)=\sum_{j=n}^{\infty}\frac{1}{j^2}\sum_{\ell=j}^{\infty}\frac{1}{\ell^3}+\sum_{j=n}^{\infty}\frac{1}{j^3}\sum_{\ell=j+1}^{\infty}\frac{1}{\ell^2}.$$

Next,

$$\begin{aligned}\sum_{n=1}^{\infty}(2n-1)\sum_{j=n}^{\infty}\frac{1}{j^2}\sum_{\ell=j}^{\infty}\frac{1}{\ell^3}&=\sum_{j=1}^{\infty}\frac{1}{j^2}\sum_{n=1}^{j}(2n-1)\sum_{\ell=j}^{\infty}\frac{1}{\ell^3}=\sum_{j=1}^{\infty}\sum_{\ell=j}^{\infty}\frac{1}{\ell^3}\\&=\sum_{\ell=1}^{\infty}\frac{1}{\ell^3}\sum_{j=1}^{\ell}1=\sum_{\ell=1}^{\infty}\frac{1}{\ell^2}=\zeta(2),\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}(2n-1)\sum_{j=n}^{\infty}\frac{1}{j^3}\sum_{\ell=j+1}^{\infty}\frac{1}{\ell^2}&=\sum_{j=1}^{\infty}\frac{1}{j^3}\sum_{n=1}^{j}(2n-1)\sum_{\ell=j+1}^{\infty}\frac{1}{\ell^2}=\sum_{j=1}^{\infty}\frac{1}{j}\sum_{\ell=j+1}^{\infty}\frac{1}{\ell^2}\\&=\sum_{\ell=2}^{\infty}\frac{1}{\ell^2}\sum_{j=1}^{\ell-1}\frac{1}{j}=\sum_{\ell=2}^{\infty}\frac{H_\ell-\frac{1}{\ell}}{\ell^2}=\sum_{\ell=1}^{\infty}\frac{H_\ell}{\ell^2}-\sum_{\ell=1}^{\infty}\frac{1}{\ell^3}\\&=2\zeta(3)-\zeta(3)=\zeta(3),\end{aligned}$$

where $H_n=\sum_{j=1}^{n}\frac{1}{j}$ denotes the nth harmonic number, and we have used the well-known identity

$$\sum_{\ell=1}^{\infty}\frac{H_\ell}{\ell^2}=2\zeta(3).$$

Finally,

$$\sum_{n=1}^{\infty}(2n-1)\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)=\zeta(2)+\zeta(3).$$

Also solved by Narendra Bhandari, Bajura, Nepal; Paul Bracken, U. of Texas, Edinburgh; Bruce Burdick, Providence, RI; Hongwei Chen, Christopher Newport U.; Eugene Herman, Grinnell C.; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Kee-Wai Lau, Hong Kong, China; Shing Hin Jimmy Pak; Seán Stewart, King Abdullay U. of Sci. and Tech., Thuwal, Saudi Arabia; and the proposer.

An integral of logarithms

1208. *Proposed by Marián Štofka, Slovak University of Technology, Bratislava, Slovakia.*

Prove that

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx = -\frac{5}{8}\zeta(3),$$

where as above, for a positive integer k, we have $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$.

Solution by Didier Pinchon, Toulouse, France.

Let I be the integral to evaluate. Using identity

$$\begin{aligned} \ln(1-x)\ \ln(1+x) &= \frac{1}{4}\left[(\ln(1-x)+\ln(1+x))^2 - (\ln(1-x)-\ln(1+x))^2\right] \\ &= \frac{1}{4}\left[\ln^2(1-x^2) - \ln^2\left(\frac{1-x}{1+x}\right)\right], \end{aligned}$$

it follows that $I = (I_1 - I_2)/4$, with

$$I_1 = \int_0^1 \frac{\ln^2(1-x^2)}{x}\,dx, \quad I_2 = \int_0^1 \frac{\ln^2\left(\frac{1-x}{1+x}\right)}{x}\,dx.$$

The substitutions $x = \sqrt{u}$ in I_1 and $x = (1-u)/(1+u)$ in I_2 give

$$I_1 = \frac{1}{2}\int_0^1 \int_0^1 \frac{\ln^2(u)}{1-u}\,du, \quad I_2 = 2\int_0^1 \int_0^1 \frac{\ln^2(u)}{1-u^2}\,du.$$

The dominated convergence theorem allows to permute the series expansion of $1/(1-u)$ (resp. $1/(1-u^2)$) with the integration in I_1 (resp. I_2), and therefore

$$I_1 = \frac{1}{2}\sum_{n\geq 0}\int_0^1 \ln^2(u)\,u^n\,du, \quad I_2 = 2\sum_{n\geq 0}\int_0^1 \ln^2(u)\,u^{2n}\,du.$$

For any nonnegative integer k, two successive integrations by parts provide the result

$$\int_0^1 \ln^2(u)\,u^n\,du = \frac{2}{(n+1)^3},$$

and it follows that

$$I_1 = \sum_{n\geq 0}\frac{1}{(n+1)^3} = \zeta(3),$$

$$I_2 = 4\sum_{n\geq 0}\frac{1}{(2n+1)^3} = 4\left[\sum_{n\geq 0}\frac{1}{(n+1)^3} - \sum_{n\geq 0}\frac{1}{(2n+2)^3}\right] = \frac{7}{2}\,\zeta(3).$$

In conclusion, $I = \frac{1}{4}\,(I_1 - I_2) = -\frac{5}{8}\,\zeta(3)$.

Several solvers pointed out that this problem, by a different proposer, appeared as problem 12256 in *The American Mathematical Monthly*.

Also solved by F. R. Ataev, Uzbekistan; Khristo Boyadzhiev, Ohio Northern U.; Brian Bradie,Christopher Newport U.; Bruce Burdick, Providence, RI; Hongwei Chen, Christopher Newport U.; Kyle Gatesman (student), Johns Hopkins U.; Subhankar Gayen, West Bengal, India; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Moubinool Omarjee, Lycée Henri IV, Paris, France; Henry Ricardo, Westchester Area Math Circle; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, King Abdullay U. of Sci. and Tech., Thuwal, Saudi Arabia; Michael Vowe, Therwil, Switzerland; and the proposer. One incomplete solution was received.

The rank of a matrix

1209. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*
For non-negative integers i and j, define

$$a_{ij} := \begin{cases} i(i-1)\cdots(i-j+1) & \text{if } 1 \le j \le i, \\ 1 & \text{if } i = 0 \text{ and } j \ge 0, \text{ or } j = 0 \text{ and } i \ge 0, \text{ and} \\ 0 & \text{if } j > i \ge 1. \end{cases}$$

Now let m be a positive integer. Prove that every $m \times m$ submatrix of the infinite matrix $(a_{2i,j})$ with $0 \le j \le m-1$ and $i \ge 0$ has rank m and, in addition, that $\sum_{i=0}^{m}(-1)^i\binom{m}{i}a_{2k+2i,j} = 0$ for $0 \le j \le m-1$ and any $k \in \mathbb{N}$.

Solution by the proposer.

Introduce the polynomials

$$f_0(x) = 1,\ f_1(x) = x,\ f_2(x) = x(x-1), \ldots,\ f_j(x) = x(x-1)\cdots(x-j+1).$$

Then $a_{2i,j} = f_j(2i)$. Since for any j

$$x^j = f_j(x) + \sum_{n=0}^{j-1} c_n f_n(x)$$

for some constants c_n, it is clear that any matrix of the form

$$\big(f_j(x_i)\big) \text{ with } 0 \le j, i \le m-1, \text{ and where all } x_i \text{ are distinct,}$$

can be transformed into a Vandermonde matrix by elementary row operations, so its determinant must be different from zero.

For the second statement, start by observing that the identity

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}f(i) = 0$$

must be valid whenever $f(x)$ is a polynomial of degree at most $m-1$. Indeed, let us define $\Delta(f(x)) = f(x) - f(x+1)$, and note that

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}f(i) = \Delta^m(f(x))(0).$$

The difference operator decreases the degree of the polynomial, and the equation can be proved inductively, using Pascal's identity.

As we saw above, the function $i \to a_{2i,j}$ is a polynomial of degree j. Hence

$$\sum_{i=0}^{m}(-1)^i \binom{m}{i} a_{2k+2i,j} = 0 \text{ for } 0 \le j \le m-1.$$

This completes the solution.

No other solutions were received.

The existence of a countable commutative integral domain with a sum-free collection of ideals

1210. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a commutative ring with identity, and let I and J be ideals of R. Recall that the *sum* of I and J is the ideal defined by $I + J := \{i + j : i \in I, j \in J\}$. Prove or disprove: there exists a countable commutative integral domain D with identity and a collection $\mathcal{S}$ of $2^{\aleph_0}$ ideals of D such that for all $I \neq J$ in $\mathcal{S}$, we have $I + J \notin \mathcal{S}$.

Solution by Anthony Bevelacqua, University of North Dakota.

Let $D = \mathbb{Z}[x_1, x_2, \ldots]$ be the polynomial ring in countably many indeterminates with coefficients in $\mathbb{Z}$. Since D is the countable union of the countable $\mathbb{Z}[x_1, \ldots, x_n]$ for each $n \in \mathbb{N}$, D is a countable commutative integral domain with identity.

For any $A \subseteq \mathbb{N}$ let I_A be the ideal of D generated by $\{x_i \mid i \in A\}$. For all $A, B \subseteq \mathbb{N}$ we have (i) $I_A = I_B$ if and only if $A = B$ and (ii) $I_A + I_B = I_{A \cup B}$. Thus it suffices to find a collection S of $2^{\aleph_0}$ subsets of $\mathbb{N}$ such that for all $A \neq B$ in S we have $A \cup B \notin S$.

It's well-known (see below for sketch of proof) that for any countable set X there exists a collection T of $2^{\aleph_0}$ subsets of X such that each $U \in T$ is infinite and for all $U \neq V$ in T we have $U \cap V$ is finite. Since each element of T is an infinite set, $U \cap V \notin T$. So there exists T a family of $2^{\aleph_0}$ subsets of $\mathbb{N}$ such that for all $U \neq V$ in T we have $U \cap V \notin T$. Now $S = \{\mathbb{N} - U \mid U \in T\}$ has the desired properties: S is a family of $2^{\aleph_0}$ subsets of $\mathbb{N}$ such that for all $A \neq B$ in S we have $A \cup B \notin S$.

Thus $D = \mathbb{Z}[x_1, x_2, \ldots]$ is a countable commutative integral domain with identity containing a collection $\mathcal{S} = \{I_A \mid A \in S\}$ of $2^{\aleph_0}$ ideals such that for all $I \neq J$ in S we have $I + J \notin \mathcal{S}$.

Sketch of a standard proof of above claim: Without loss of generality we can suppose $X = \mathbb{Q}$. There are $2^{\aleph_0}$ real irrational numbers. For each real irrational r let $(u_n)_{n=1}^{\infty}$ be a sequence of rational numbers converging to r, and let $U_r = \{u_n \mid n \in \mathbb{N}\}$. Each U_r is infinite and $U_r \cap U_s$ is finite for any distinct real irrationals r and s.

Also solved by Northwestern U. Math Problem Solving Group; and the proposer.

Correction: In the featured solution to problem 1195 in the January 2022 issue, two numerators were missing in the second line. The second line as provided by the solver should have been

$$\sum_{n=1}^{\infty}\sum_{k=n+2}^{\infty}\frac{h_n}{(n+1)k^2} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{h_n}{(n+1)(n+k+1)^2}.$$

The editor apologizes for the error.