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The Playground

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THE PLAYGROUND

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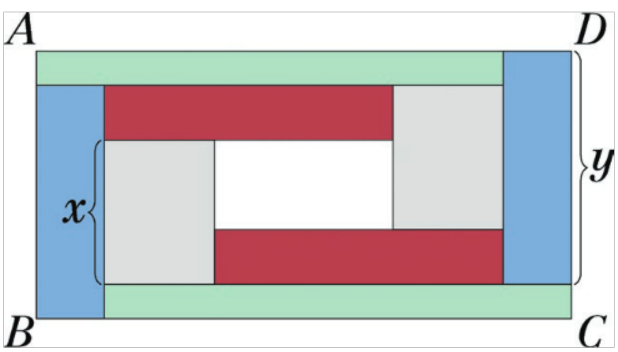
THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Nine Rectangles (P442). Arsalan Wares (Valdosta State University) posed this problem. Figure 1 shows a rectangle $ABCD$ that is composed of nine rectangles, each of the same area. Furthermore, rectangles of the same color are congruent to each other. Suppose the lengths of the gray and blue rectangles are x and y, respectively. Find x/y.

Figure 1. A rectangle composed of nine rectangles.



THE MONKEY BARS

These open-ended problems don't have a previously known exact solution, so we intend for readers to fool around with them. The Playground will publish the best submissions received (proofs encouraged but not required).

Perfect Matching Challenge (P443). William Jones (PMP) suggested this twist on a classic problem. A game show has 100 contestants, each issued a shirt with a number from 1 to 100 on it, all different. Additionally, 100 slips of paper and 100 boxes are numbered 1 to 100. The slips are mixed up and randomly distributed with one slip per box. Then, the boxes are placed in numeric order in a room separate and out of view from the contestants.

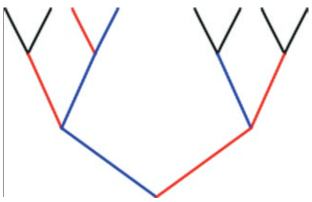
Each contestant enters the box room in turn individually and may open up to 50 boxes and examine their contents. Before exiting, each contestant has the option to swap two slips from the boxes examined. Contestants may only communicate prior to entering the box room. To win the challenge, each contestant must open the box containing the slip that matches their shirt number. What strategy should the contestants enact to maximize their chance of winning? Under this strategy, what is the probability of succeeding?

THE ZIP LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip-Line problems require you to read the corresponding article, but doing so can never hurt, of course.

Tic-Tac-Tree (P444). Parker Glynn-Adey (University of Toronto) provided this problem connected to his article "A *Potential* Strategy for Huge Tic-Tac-Toe" (page 16). *Tic-Tac-Tree* is a variant of Tic-Tac-Toe played by two players, Red and Blue. The players alternate coloring the edges of a rooted tree. By convention, Red plays first. If a player colors a path from top to bottom, then the player wins. Consider the board consisting of a tree with a root at the bottom and branches upward in

Figure 2. An example of a Tic-Tac-Tree game position.



pairs at each level. The position shown in figure 2 shows a board of height three with a win for Blue. Does the Erdős-Selfridge argument apply to Tic-Tac-Tree played on a board of height n? If both players play optimally, what is the outcome?

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

Complete the Identity (P445). Danesh Forouhari (San Francisco Bay Area) asks us to consider the Virahāṅka-Fibonacci sequence (F_n). Assume the units digit of F_n and F_{n+2} both are 7. What is the units digit of $F_n^5 + F_{n+1}^5 + F_{n+2}^5 + F_{n+3}^5$?

THE CAROUSEL—OLDIES, BUT GOODIES

In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

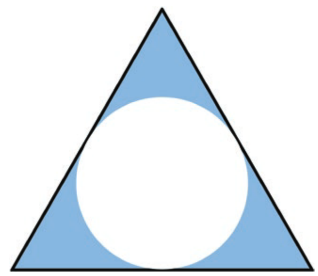
Preferred Parking (C39). Assume a one-way street has n parking spots, labeled consecutively from 1 to n. A sequence of n drivers will arrive at the street looking to park. Each driver proceeds to their preferred space and parks if the space is vacant. Otherwise, the driver continues down the one-way street and parks in the first open space. If no space is available, then the driver is unable to park. How many sequences of preferences of n drivers result in all drivers being able to park?

APRIL WRAP-UP

Holes Uncovered (P423). Maintenance holes are round supposedly so that covers can't drop into the holes. Of course, even a round hole must have some "lip" (area filled in to reduce the aperture) to prevent the cover from falling in. However, any reduction to the hole diameter suffices, so the infimum of lip areas that will work is zero.

Suppose we only allow lips that leave a single convex aperture. What's the infimum lip area that will prevent the cover of a hole in the shape of an equilateral triangle from falling in? Note that the remaining aperture is *not* required to be a triangle—it need only prevent the equilateral triangular cover from passing through in any orientation (see figure 3 for an example).

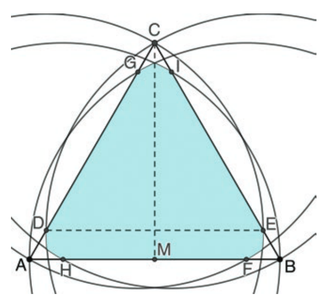
Figure 3. The (non-optimal) blue shaded lip will prevent the cover of the equilateral triangular hole outlined in black from falling through.



A PMP member from Texas proposes the following lip of very small area; can anyone find a better lip or a proof that this one is optimal?

We wish to remove some portion of equilateral triangle ABC with unit sides so that the remaining convex region contains no line segment longer than its height CM, the shortest distance between two parallel lines enclosing ABC. Select D on AC and E on BC so that DE is parallel to AB and has length CM, and similarly construct segments FG and HI. Then, as shown in figure 4, remove the intersection of the interiors of circles of radius CM centered on each of the six points D through I, keeping the remainder as the lip. Evidently the remaining aperture cannot pass the triangular cover, and the submitter calculates the removed area as no more than 0.01606.

Figure 4. A region of diameter CM inside ABC.



Candy Circle (P428). This problem follows up on one in Florentino and Higginbottom's article "Three-Pile Candy Sharing." Three students start with piles of $2j$, $2j \pm 2$, and $2j + 2k$ candies, respectively, for integers $j > 1$ and $k > -j$. In each round, all the students give each of the others half of their pile, and then receive (from an external supply) one additional candy if they happen to end up with an odd number of candies.

Show that after a finite number of rounds, all students have the same number of candies, and determine how many rounds it will take to reach this state. Can you determine the final number of candies each student will have at the end?

Although this problem originally ran a year ago, we just received solutions from Hyunbin Yoo of South Korea and a PMP participant in Texas. First, the symmetry of this formulation of the game makes the positions of the players and whether one player has two more or two less than another irrelevant. That is, by reassigning positions and adjusting the value of j, we may assume that the game starts with $2j$, $2j + 2$, and $2j + 2k$ candies, where $j \geq 0$ and $k \geq -j$. Let's call a state of this form (possibly after reordering the participants and reassigning j and/or k) a k-state of the game.

The redistribution step leads to $2j + k + 1$, $2j + k$, and $2j + 1$ candies. If k is odd, the replenishment step produces the two-of-a-kind state $2j + k + 1$, $2j + k + 1$, $2j + 2$ with parameter $(k - 1) / 2$. This always terminates (in $N((k - 1) / 2)$ rounds for a function N given in the article) ending up with

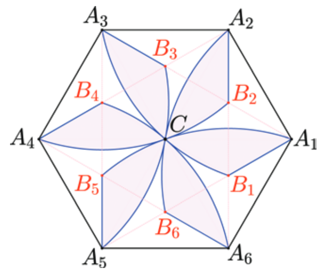
$$2j+1+\frac{1}{3}\left(k+2d\left(\frac{k-1}{2}\right)\right)$$

candies for each player, where $d(n)$ is the number of ones in the binary representation of n.

If k is even, however, replenishment yields $2j + k + 2$, $2j + k$, $2j + 2$ candies. Here we again have a difference of two between two players, and this turns out to be a $(1 - k / 2)$-state. Therefore, the absolute value of the state has decreased. So, after a finite number of turns, a two-of-a-kind state must occur (because it surely does when k is 0 or 1), and the game must terminate. Further, $1 - k / 2$ is odd or even according to whether the twos digit of k in binary is 0 or 1, respectively. Thus, the game remains in an "off-by-2" ℓ-state for some ℓ for each consecutive binary 1 (from the twos position) in k; call the number of these $o(k)$. After $o(k)$ rounds, the game reverts to a two-of-a-kind state, and the subtraction of 1 from $k / 2$ in that step produces as many ones as the number $z(k)$ of consecutive zeros in the binary representation of k to the left of the block of $o(k)$ ones. Because every round while off-by-2 adds two candies, the total number of candies added is $2d(k) + 2z(k) + 2$, from which it is straightforward to compute the equilibrium state, and the number of turns played is $o(k) + N(m)$ where m is the parameter of the first two-of-a-kind state encountered.

Flower Fraction (P434). Arsalan Wares provided this lovely geometric problem. You are given a regular hexagon h, centered at point C, with vertices A_1 through A_6 in counterclockwise order, as shown in figure 5. (We interpret subscripts other than 1,…,6 cyclically, so that A_7 again refers to A_1 and A_0 to A_6, etc.) Let point B_i be the intersection of diagonals $A_{i-2}A_i$ and $A_{i-1}A_{i+1}$. Also, let $\overset{\frown}{A_iC}$ and $\overset{\frown}{CB_{i-2}}$ refer to the corresponding arcs of the circles determined by points A_i, C, and B_{i-2}.

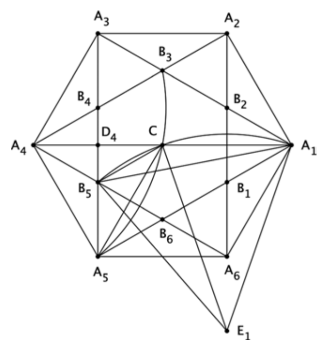
Figure 5. A flower inscribed in a hexagon.



Shade the six disjoint regions each bounded by B_iA_i, $\overset{\frown}{A_iC}$, and $\overset{\frown}{CB_i}$ for i from 1 to 6 to create a floral pattern inscribed in hexagon h. What fraction of the area of h has been shaded?

These petals attracted the attention of John Cade (of Pikeville, Kentucky, whose solution we follow below), Dmitry Fleischman, Randy K. Schwartz (Schoolcraft College), and the problem-solving group at Georgia Southern University.

Figure 6. Determining the area of petal A_5CB_5.



It's easiest to set the edge length of the hexagon as one unit and compute the area of one petal divided by the area of a unit-side equilateral triangle ($\sqrt{3} / 4$). Decompose the lower-left petal as shown in figure 6 into triangle A_5CB_5 (one third of an equilateral, with area $\sqrt{3} / 12$) and circular segments B_5C and CA_5, the latter of which is congruent to circular segment CA_1. Thus the remaining area of the petal is the area of the 1/6-sector of the circle bounded by B_5E_1, E_1A_1 and arc A_1CB_5 minus the areas of isosceles triangles CB_5E_1 and A_1CE_1. The Pythagorean theorem yields that the radius of this sector, which is equal to A_1B_5, is $\sqrt{7/3}$. Hence, it has area $7\tau / 36$.

The side lengths of the two isosceles triangles are straightforward to determine (two sides of each are radii of the arc and $A_1C = 1$, which just leaves the calculation of $B_5C = \sqrt{3} / 3$), so using Heron's formula we find that the triangular areas are $\sqrt{3} / 4$ and $5\sqrt{3} / 12$, respectively. Putting all these ingredients together, the desired ratio of flower to hexagon is

$$\left(\frac{\sqrt{3}}{12}+\frac{7\tau}{18}-\frac{\sqrt{3}}{4}-\frac{5\sqrt{3}}{12}\right)\frac{4}{\sqrt{3}}=\frac{7}{27}(\sqrt{3}\tau-9)\approx 0.4881.$$

Sumful Sets (P435). This problem was submitted by Veit Elser (Cornell University). Consider only natural numbers (nonnegative integers).

1) Find two infinite sets of numbers such that every number can be uniquely expressed as the sum of two numbers, one from each set.
2) Discounting swapping the two sets, is your solution unique?
3) For each $n > 2$, find a collection of n infinite sets so that every number can be uniquely expressed as a sum of n numbers, one from each set.
4) Is it possible to find a collection of $n \geq 2$ pairwise disjoint infinite sets of numbers so that every sufficiently large number can be uniquely expressed as a sum of n numbers, one from each set?

Matthew Helmer (Pacific Lutheran University) and William Hu (Harvard University) submitted solutions to parts 1–3 of this problem. For the first part, you can take the sets of all sums of odd powers of two and all sums of even powers of two. Because every number can be uniquely written as a sum of powers of two, the desired decomposition follows. This solution is not unique as we clearly get another by simply choosing a different partition of the exponents of two. Moreover, we can obtain n-set solutions by taking an n-part partition of the exponents.

The striking feature of all these and all solutions similar in concept is that every set of summands contains zero. So all parts of all submitted solutions intersected at zero. On the other hand, no arguments were offered why such sets of summands couldn't be disjoint. Any evidence on this question (i.e., part 4 of the original problem) either way would be welcome.

Circellipse? (P436). We received no solutions to this problem. An interested reader should be able to construct a solution from the information found at *https://en.wikipedia.org/wiki/Steiner_chain*.

Cubarc Sine (P437) (this problem erroneously appeared under the already-used title "Exponential Harmony."**)**. Goran Conar (Varaždin, Croatia) asked us to consider positive real numbers x, y, and z such that $x + y + z = 1$, and to show that

$$x^3 + y^3 + z^3 \leq \sin(x \arcsin x^2 + y \arcsin y^2 + z \arcsin z^2).$$

Under what conditions does equality occur?

We received the following solution from Henry Ricardo (Westchester Area Math Circle). The conditions $x, y, z > 0$ and $x + y + z = 1$ imply $0 < x, y, z < 1$. The function $f(x) = \arcsin x$ is convex on the interval (0, 1) because $f''(x) = x / (1 - x^2)^{3/2} \geq 0$. So we may apply the weighted Jensen inequality:

$$\arcsin\left(\frac{x \cdot x^2 + y \cdot y^2 + z \cdot z^2}{x + y + z}\right) \leq \frac{x \arcsin x^2 + y \arcsin y^2 + z \arcsin z^2}{x + y + z},$$

from which we can drop the denominator on both sides as it is equal to 1 by hypothesis.

As the sine function is increasing on the interval $[0, \tau / 4] \supset (0, 1)$, we can apply it to both sides, precisely yielding the desired inequality. Equality holds if and only if $x = y = z = 1 / 3$.

CAROUSEL SOLUTION

The following elegant solution is attributed to Henry O. Pollak. Instead of n spots on a one-way street, consider $n + 1$ parking spaces arranged in a circle. Then suppose drivers are allowed to express a preference for any of the $n + 1$ spaces. There are $(n + 1)^n$ sequences of preferences. Each preference sequence results in all n drivers finding a parking space on the augmented street. If the preference sequence contains a value of $n + 1$, then spot $n + 1$ is occupied. When all n drivers find a parking space on the original one-way street, then the $n + 1$ space is vacant. If not, the $n + 1$ space is occupied. Because there are n drivers and $n + 1$ spaces, each preference sequence results in a vacant space. From this viewpoint, all parking spaces are treated the same. Thus, each vacant space appears in $\frac{1}{n+1}$ of the preferences sequences. It follows that there are $\frac{1}{n+1}(n + 1)^n = (n + 1)^{n-1}$ sequences with spot $n + 1$ vacant as needed.

SUBMISSION AND CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems and solutions should be submitted to *MHproblems@maa.org* and *MHsolutions@maa.org*, respectively (PDF format preferred). Paper submissions can be sent to Jeremiah Bartz, UND Math Dept., Witmer Hall 313, 101 Cornell St. Stop 8376, Grand Forks, ND 58202-8376. Please include your name, email address, and affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is January 15, 2023.

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