THE CHINESE TEACHERS' APPROACH TO THE MEANING OF DIVISION BY FRACTIONS

The deficiency in the subject matter knowledge of the U.S. teachers on the advanced arithmetical topic of division by fractions did not appear among the Chinese teachers. While only one among the 23 U.S. teachers generated a conceptually correct representation for the meaning of the equation, 90% of the Chinese teachers did. Sixty-five of the 72 Chinese teachers created a total of more than 80 story problems representing the meaning of division by a fraction. Twelve teachers proposed more than one story to approach different aspects of the meaning of the operation. Only six (8%) teachers said that they were not able to create a story problem, and one teacher provided an incorrect story (which represented $\frac{1}{2} \div 1\frac{3}{4}$ rather than $1\frac{3}{4} \div \frac{1}{2}$). Figure 3.1 displays a comparison of teachers' knowledge about this topic.

The Chinese teachers represented the concept using three different models of division: measurement (or quotitive), partitive, and product and factors.[6] For example, $1\frac{3}{4} \div \frac{1}{2}$ might represent:

- $1\frac{3}{4}$ feet $\div \frac{1}{2}$ feet $= \frac{7}{2}$ (measurement model)
- $1\frac{3}{4}$ feet $\div \frac{1}{2} = \frac{7}{2}$ feet (partitive model)
- $1\frac{3}{4}$ square feet $\div \frac{1}{2}$ feet $= \frac{7}{2}$ feet (product and factors)

which might correspond to:

- How many $\frac{1}{2}$-foot lengths are there in something that is 1 and $\frac{3}{4}$ feet long?
- If half a length is 1 and $\frac{3}{4}$ feet, how long is the whole?
- If one side of a $1\frac{3}{4}$ square foot rectangle is $\frac{1}{2}$ feet, how long is the other side?

The Models of Division by Fractions

The Measurement Model of Division: "Finding How Many $\frac{1}{2}$s There Are in $1\frac{3}{4}$" or "Finding How Many Times $1\frac{3}{4}$ is of $\frac{1}{2}$"

Sixteen stories generated by the teachers illustrated two ideas related to the measurement model of division: "finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$" and "finding how many times $1\frac{3}{4}$ is of $\frac{1}{2}$." Eight stories about five topics corresponded to "finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$." Here are two examples:

[6]Greer (1992) gives an extensive discussion of models of multiplication and division. His category "rectangular area" is included in "product and factors."

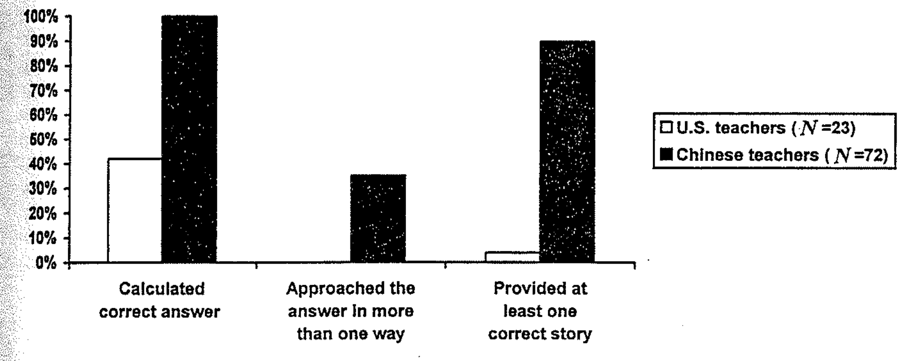

FIG. 3.1. Teachers' knowledge of division by fractions.

> Illustrating it with the measurement model of division, $1\frac{3}{4} \div \frac{1}{2}$ can be articulated as how many $\frac{1}{2}$s there are in $1\frac{3}{4}$. To represent it we can say, for example, given that a team of workers construct $\frac{1}{2}$ km of road each day, how many days will it take them to construct a road of $1\frac{3}{4}$ km long? The problem here is to find how many pieces of $\frac{1}{2}$ km, which they can accomplish each day, are contained in $1\frac{3}{4}$ km. You divide $1\frac{3}{4}$ by $\frac{1}{2}$ and the result is $3\frac{1}{2}$ days. It will take them $3\frac{1}{2}$ days to construct the road. (Tr. R.)

> Cut an apple into four pieces evenly. Get three pieces and put them together with a whole apple. Given that $\frac{1}{2}$ apple will be a serving, how many servings can we get from the $1\frac{3}{4}$ apples? (Ms. I.)

"Finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$" parallels the approach of Tr. Belle, the U.S. teacher who had a conceptual understanding of the topic. There were eight other stories that represented "finding how many times $1\frac{3}{4}$ is of $\frac{1}{2}$." For example:

> It was planned to spend $1\frac{3}{4}$ months to construct a bridge. But actually it only took $\frac{1}{2}$ month. How many times is the time that was planned of the time that actually was taken? (Tr. K.)

"Finding how many $\frac{1}{2}$s there are in $1\frac{3}{4}$" and "finding how many times $1\frac{3}{4}$ is of $\frac{1}{2}$" are two approaches to the measurement model of division by fractions. Tr. Li indicated that though the measurement model is consistent for whole numbers and fractions when fractions are introduced the model needs to be revised:

> In whole number division we have a model of finding how many times one number is of another number. For example, how many times the number 10 is of the number 2? We divide 10 by 2 and get 5. 10 is 5 times 2. This is what we call the measurement model. With fractions, we can still say, for example, what times $\frac{1}{2}$ is $1\frac{3}{4}$? Making a story problem, we can say for instance, there are two fields. Field A is $1\frac{3}{4}$ hectares, and field B is $\frac{1}{2}$ hectare. What

times the area of field B is the area of field A? To calculate the problem we divide $1\frac{3}{4}$ hectares by $\frac{1}{2}$ hectare and get $3\frac{1}{2}$. Then we know that the area of the field A is $3\frac{1}{2}$ times that of the field B. The equation you asked me to represent fits this model. However, when fractions are used this division model of measurement need to be revised. In particular, when the dividend is smaller than the divisor and then the quotient becomes a proper fraction. Then the model should be revised. The statement of "finding what fraction one number is of another number," or, "finding what fractional times one number is of another number" should be added on the original statement. For example, for the expression $2 \div 10$, we may ask, what fraction of 10 is 2? Or, what fractional times is 2 of 10? We divide 2 by 10 and get $\frac{1}{5}$. 2 is $\frac{1}{5}$ of 10. Similarly, we can also ask: What is the fractional part that $\frac{1}{4}$ is of $1\frac{1}{2}$? Then you should divide $\frac{1}{4}$ by $1\frac{1}{2}$ and get $\frac{1}{6}$.

The Partitive Model of Division: Finding a Number Such That $\frac{1}{2}$ of It is $1\frac{3}{4}$

Among more than 80 story problems representing the meaning of $1\frac{3}{4} \div \frac{1}{2}$, 62 stories represented the partitive model of division by fractions—"finding a number such that $\frac{1}{2}$ of it is $1\frac{3}{4}$":

> Division is the inverse of multiplication. Multiplying by a fraction means that we know a number that represents a whole and want to find a number that represents a certain fraction of that. For example, given that we want to know what number represents $\frac{1}{2}$ of $1\frac{3}{4}$, we multiply $1\frac{3}{4}$ by $\frac{1}{2}$ and get $\frac{7}{8}$. In other words, the whole is $1\frac{3}{4}$, and $\frac{1}{2}$ of it is $\frac{7}{8}$. In division by a fraction, on the other hand, the number that represents the whole becomes the unknown to be found. We know a fractional part of it and want to find the number that represents the whole. For example, $\frac{1}{2}$ of a jump-rope is $1\frac{3}{4}$ meters, what is the length of the whole rope? We know that a part of a rope is $1\frac{3}{4}$ meters, and we also know that this part is $\frac{1}{2}$ of the rope. We divide the number of the part, $1\frac{3}{4}$ meters, by the corresponding fraction of the whole, $\frac{1}{2}$, we get the number representing the whole, $3\frac{1}{2}$ meters. Dividing $1\frac{3}{4}$ by $\frac{1}{2}$, we will find that the whole rope is $3\frac{1}{2}$ meters long . . . But I prefer not to use dividing by $\frac{1}{2}$ to illustrate the meaning of division by fractions. Because one can easily see the answer without really doing division by fractions. If we say $\frac{4}{5}$ of a jump-rope is $1\frac{3}{4}$ meters, how long is the whole rope? The division operation will be more significant because then you can't see the answer immediately. The best way to calculate it is to divide $1\frac{3}{4}$ by $\frac{4}{5}$ and get $2\frac{3}{16}$ meters. (Ms. G.)

> Dividing by a fraction is finding a number when a fractional part of it is known. For example, given that we know that $\frac{1}{2}$ of a number is $1\frac{3}{4}$, dividing $1\frac{3}{4}$ by $\frac{1}{2}$, we can find out that this number is $3\frac{1}{2}$. Making a story problem to illustrate this model, let's say that one kind of wood weighs $1\frac{3}{4}$ tons per cubic meter, it is just $\frac{1}{2}$ of the weight of per cubic meter of one kind of marble. How much does one cubic meter of the marble weigh? So we know that $\frac{1}{2}$ cubic meter of the marble weighs $1\frac{3}{4}$ tons. To find the weight of one cubic meter of it, we divide $1\frac{3}{4}$, the number that represents the fractional part, by $\frac{1}{2}$, the fraction which $1\frac{3}{4}$ represents, and get $3\frac{1}{2}$, the number of the whole. Per cubic meter the marble weighs $3\frac{1}{2}$ tons. (Tr. D.)

> My story will be: A train goes back and forth between two stations. From Station A to Station B is uphill and from Station B back to Station A is downhill. The train takes $1\frac{3}{4}$ hours going from Station B to Station A. It is only $\frac{1}{2}$ time of that from Station A to Station B. How long does the train take going from Station A to Station B? (Tr. S.)

> The mom bought a box of candy. She gave $\frac{1}{2}$ of it which weighed $1\frac{3}{4}$ kg to the grandma. How much did the box of the candy originally weigh? (Ms. M.)

The teachers above explained the fractional version of the partitive model of division. Tr. Mao discussed in particular how the partitive model of division by integers is revised when fractions are introduced:

> With integers students have learned the partitive model of division. It is a model of finding the size of each of the equal groups that have been formed from a given quantity. For example, in our class we have 48 students, they have been formed into 4 groups of equal size, how many students are there in each group? Here we know the quantity of several groups, 48 students. We also know the number of groups, 4. What to be found is the size of one group. So, *a partitive model is finding the value of a unit when the value of several units is known.* In division by fractions, however, the condition has been changed. Now what is known is not the value of several units, rather, the value of a part of the unit. For example, given that we paid $1\frac{3}{4}$ Yuan to buy $\frac{1}{2}$ of a cake, how much would a whole cake cost? Since we know that $\frac{1}{2}$ of the whole price is $1\frac{3}{4}$ Yuan, to know the whole price we divide $1\frac{3}{4}$ by $\frac{1}{2}$ and get $3\frac{1}{2}$ Yuan. In other words, *the fractional version of the partitive model is to find a number when a part of it is known.* (italics added)

Tr. Mao's observation was true. Finding a number when several units is known and finding a number when a fractional part of it is known are represented by a common model—finding the number that represents a unit when a certain amount of the unit is known. What differs is the feature of the amount: with a whole number divisor, the condition is that "*several times* the unit is known," but with a fractional divisor the condition is that "*a fraction* of the unit is known." Therefore, conceptually, these two approaches are identical.

This change in meaning is particular to the partitive model. In the measurement model and the factors and product model, division by fractions keeps the same meaning as whole number division. This may explain why so many of the Chinese teachers' representations were partitive.

Factors and Product: Finding a Factor That Multiplied by $\frac{1}{2}$ Will Make $1\frac{3}{4}$

Three teachers described a more general model of division—to find a factor when the product and another factor are known. The teachers articulated it as "to find a factor that when multiplied by $\frac{1}{2}$ makes $1\frac{3}{4}$":

> As the inverse operation of multiplication, division is to find the number representing a factor when the product and the other factor are known. From this perspective, we can get a word problem like "Given that the product of $\frac{1}{2}$ and another factor is $1\frac{3}{4}$, what is the other factor?" (Tr. M.)

> We know that the area of a rectangle is the product of length and width. Let's say that the area of a rectangle board is $1\frac{3}{4}$ square meters, its width is $\frac{1}{2}$ meters, what is its length? (Mr. A.)

These teachers regarded the relationship between multiplication and division in a more abstract way. They ignored the particular meaning of the multiplicand and multiplier in multiplication and related models of division. Rather, they perceived the multiplicand and multiplier as two factors with the same status. Their perspective, indeed, was legitimized by the commutative property of multiplication.

The concept of fractions as well as the operations with fractions taught in China and U.S. seem different. U.S. teachers tend to deal with "real" and "concrete" wholes (usually circular or rectangular shapes) and their fractions. Although Chinese teachers also use these shapes when they introduce the concept of a fraction, when they teach operations with fractions they tend to use "abstract" and "invisible" wholes (e.g., the length of a particular stretch of road, the length of time it takes to complete a task, the number of pages in a book).

Meaning of Multiplication by a Fraction: The Important Piece in the Knowledge Package

Through discussion of the meaning of division by fractions, the teachers mentioned several concepts that they considered as pieces of the knowledge package related to the topic: the meaning of whole number multiplication, the concept of division as the inverse of multiplication, models of whole number division, the meaning of multiplication with fractions, the concept of a fraction, the concept of a unit, etc. Figure 3.2 gives an outline of the relationships among these items.

The learning of mathematical concepts is not a unidirectional journey. Even though the concept of division by fractions is logically built on the previous learning of various concepts, it, in turn, plays a role in reinforcing and deepening that previous learning. For example, work on the meaning

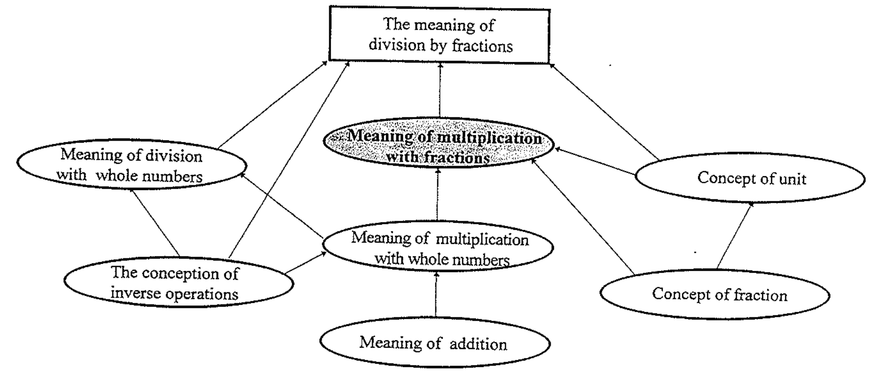

FIG. 3.2. A knowledge package for understanding the meaning of division by fractions.

of division by fractions will intensify previous concepts of rational number multiplication. Similarly, by developing rational number versions of the two division models, one's original understanding of the two whole number models will become more comprehensive:

> This is what is called "gaining new insights through reviewing old ones." The current learning is supported by, but also deepens, the previous learning. The meaning of division by fractions seems complicated because it is built on several concepts. On the other hand, however, it provides a good opportunity for students to deepen their previous learning of these concepts. I am pretty sure that after approaching the meaning and the models of division by fractions, students' previous learning of these supporting concepts will be more comprehensive than before. Learning is a back and forth procedure. (Tr. Sun)

From this perspective, learning is a continual process during which new knowledge is supported by previous knowledge and the previous knowledge is reinforced and deepened by new knowledge.

During the interviews, "the meaning of multiplication with fractions" was considered a key piece of the knowledge package. Most teachers considered multiplication with fractions the "necessary basis" for understanding the meaning of division by fractions:

> The meaning of multiplication with fractions is particularly important because it is where the concepts of division by fractions are derived . . . Given that our students understand very well that multiplying by a fraction means finding a fractional part of a unit, they will follow this logic to understand how the models of its inverse operation work. On the other hand, given that they do not have a clear idea of what multiplication with fractions means, concepts of division by a fraction will be arbitrary for them and very difficult to understand. Therefore, in order to let our students grasp the

meaning of division by fractions, we should first of all devote significant time and effort when teaching multiplication with fractions to make sure students understand thoroughly the meaning of this operation . . . Usually, my teaching of the meaning of division by fractions starts with a review of the meaning of multiplication with fractions. (Tr. Xie)

The concepts of division by fractions, such as "finding a number when a fractional part is known" or "finding what fraction one number is of another number," etc. sound complicated. But once one has a comprehensive understanding of the meaning of multiplication with fractions, one will find that these concepts are logical and easy to understand. Therefore, to help students to understand the meaning of division by fractions, many of our efforts are not devoted directly to the topic, but rather, to their thorough understanding of the meaning of multiplication with fractions, and the relationship between division and multiplication. (Tr. Wu)

The meaning of multiplication with fractions is also important in the knowledge package because it "connects several relevant conceptions":

The concept of multiplication with fractions is like a "knot." It "ties" several other important concepts together. As the operation of multiplication, it is connected with concepts of whole number addition and division. Moreover, in the sense that it deals with fractional numbers, it is related to the conception of a fraction, and those of addition and division with fractions. A grasp of the meaning of multiplication with fractions depends on comprehension of several concepts. At the same time, it substantially reinforces one's previous learning and contributes to one's future learning. (Ms. I.)

Indeed, from the teachers' perspective, the importance of pieces of knowledge in mathematics is not the same. Some of them "weigh" more than others because they are more significant to students' mathematical learning. In addition to "the power of supporting" that we have discussed earlier, another aspect that contributes to the importance of a piece of knowledge is its "location" in a knowledge network. For example, multiplication with fractions is important also because it is at an "intersection" of several mathematical concepts.

The Representations of the Models of Division by Fractions

The Chinese teachers' profound understanding of the meaning of division by fractions and its connections to other models in mathematics provided them with a solid base on which to build their pedagogical content knowledge of the topic. They used their vivid imaginations and referred to rich topics to represent a single concept of division by fractions. On the other hand, some teachers used one subject to generate several story problems to represent various aspects of the concept. Teachers also drew on knowledge of elementary geometry—the area of a rectangle—to represent division.

Rich Topics Representing the Partitive Model

Even though the operation of division has two models, it appears that the two models do not receive the same attention. For most of the teachers in our research, the partitive model was substantially more impressive than the measurement model. Teachers referred to about thirty subjects in generating more than sixty story problems to represent the fractional version of the partitive model of division. In addition to those discussed earlier, here are a few other examples:

A factory that produces machine tools now uses $1\frac{3}{4}$ tons of steel to make one machine tool, $\frac{1}{2}$ of what they used to use. How much steel did they used to use for producing one machine tool? (Ms. H.)

Uncle Wang ploughed $1\frac{3}{4}$ mus[7] in $\frac{1}{2}$ a day; with this speed, how many mus can he plough in a whole day? (Mr. B.)

Yesterday I rode a bicycle from town A to town B. I spent $1\frac{3}{4}$ hour for $\frac{1}{2}$ of my journey, how much time did I take for the whole journey? (Tr. R.)

A farm has $1\frac{3}{4}$ mus of experimental fields growing wheat. It is $\frac{1}{2}$ of the area of the experimental field growing cotton. How big is the field of cotton? (Tr. N.)

In a river with swift current a downriver boat takes only $\frac{1}{2}$ the time of an upriver boat to go the same long journey. Now we have a downriver boat which took $1\frac{3}{4}$ hour going from place A to place B, how long it will take an upriver boat to go from place B to place A? (Tr. Mao)

Given that we want to know how much vegetable oil there is in a big bottle, but we only have a small scale. We draw $\frac{1}{2}$ of the oil from the bottle, weigh it, and find that it is $1\frac{3}{4}$ kg. Can you tell me how much all the oil in the bottle originally weighed? (Ms. R.)

One day Xiao-Min went to downtown to see a movie. On his way he ran into his aunt. Xiao-Min asked her, "Do you know how far is it from our village to downtown?" His aunt said, "I am not going to tell you the number but I will give you a clue. You have walked $1\frac{3}{4}$ lis[8] and it is exactly $\frac{1}{2}$ of the whole distance. Figure out your question on your own." (Ms. K.)

While the U.S. teachers tended to use a concrete whole (such as round food) and its parts to represent a whole and a fraction, most Chinese teachers

[7] "Mu" is a Chinese measurement for area. Fifteen mus is one hectare.
[8] "Li" is a traditional measurement for distance. One li is $\frac{1}{2}$ kilometer.

represented these concepts in a more abstract way. Only 3 of the 72 teachers used round food as the subject of their representation. In many story problems created by the Chinese teachers, $3\frac{1}{2}$, the quotient of the division, was treated as a unit, and $1\frac{3}{4}$, the dividend, was regarded as $\frac{1}{2}$ of the unit.

While food and money were the two main subjects of U.S. teachers' representations, those used by the Chinese teachers were more diverse. In addition to topics in students' lives, those related to students' lives were also included, such as what happens in a farm, in a factory, in the family, etc. Teachers' solid knowledge of the meaning of division by fractions made them comfortable using a broad range of topics in representations.

Several Stories With a Single Subject

Among the teachers who created more than one story to illustrate various aspects of the concept of division by fractions, Ms. D. stood out. She generated three stories about the same subject:

> The equation of $1\frac{3}{4} \div \frac{1}{2} =$ can be represented from different perspectives. For instance, we can say, here is $1\frac{3}{4}$ kg sugar and we want to wrap it into packs of $\frac{1}{2}$ kg each. How many packs can we wrap? Also, we can say that here we have two packs of sugar, one of white sugar and the other of brown sugar. The white sugar is $1\frac{3}{4}$ kg and the brown sugar is $\frac{1}{2}$ kg. How many times is the weight of white sugar of that of brown sugar? Still, we can say that here is some sugar on the table that weighs $1\frac{3}{4}$ kg; it is $\frac{1}{2}$ of all the sugar we now have at home, so how much sugar do we have at home? All three stories are about sugar, and all of them represent $1\frac{3}{4} \div \frac{1}{2}$. But the numerical models they illustrate are not the same. I would put the three stories on the board and invite my students to compare the different meanings they represent. After the discussion I would ask them to try to make up their own story problems to represent the different models of division by fractions. (Ms. D.)

In order to involve students in a comparison of the different concepts associated with $1\frac{3}{4} \div \frac{1}{2}$, Ms. D. created several representations with a single subject. The similarity in the subject and the similarity in the numbers included in the operation would make the difference in the numerical models that the stories represented more obvious to students.

DISCUSSION

Calculation: How Did It Reveal Teachers' Understanding of Mathematics?

The difference between the mathematical knowledge of the U.S. teachers and that of the Chinese teachers became more striking with the topic of division by fractions. The first contrast was presented in calculation. The interview question of this chapter asked the teachers to calculate $1\frac{3}{4} \div \frac{1}{2}$. The process of calculation revealed features of teachers' procedural knowledge and of their understanding of mathematics, as well as of their attitude toward the discipline.

In the two previous chapters all teachers presented a sound procedural knowledge. This time, only 43% of the U.S. teachers succeeded in calculation and none of them showed an understanding of the rationale of the algorithm. Most of these teachers struggled. Many tended to confound the division by fractions algorithm with those for addition and subtraction or for multiplication. These teachers' procedural knowledge was not only weak in division with fractions, but also in other operations with fractions. Reporting that they were uncomfortable doing calculation with mixed numbers or improper fractions, these teachers' knowledge about the basic features of fractions was also very limited.

All of the Chinese teachers succeeded in their calculations and many of them showed enthusiasm in doing the problem. These teachers were not satisfied by just calculating and getting an answer. They enjoyed presenting various ways of doing it—using decimals, using whole numbers, applying the three basic laws, etc. They went back and forth across subsets of numbers and across different operations, added and took off parentheses, and changed the order of operations. They did this with remarkable confidence and amazingly flexible skills. In addition, many teachers made comments on various calculation methods and evaluated them. Their way of "doing mathematics" showed significant conceptual understanding.

Another interesting feature of the Chinese teachers' mathematics is that they tended to provide "proofs" for their calculation procedures. Most teachers justified their calculations by mentioning the rule that "dividing by a number is equivalent to multiplying by its reciprocal." Others converted the fraction $\frac{1}{2}$ into $1 \div 2$ and proved step by step that dividing by $\frac{1}{2}$ is equivalent to multiplying by 2. Still other teachers used the meaning of dividing by $\frac{1}{2}$ to explain the calculating procedure. Their performance is mathematician-like in the sense that to convince someone of a truth one needs to prove it, not just assert it.

"A Concept Knot": Why It is Important

In addition to their performance in "doing mathematics," the Chinese teachers showed a knowledge of fractions that was markedly more solid than that of the U.S. teachers in other ways. The Chinese teachers were aware of abundant connections between fractions and other mathematical topics. They were aware of how a fraction can be written as a division expression in which the numerator is the dividend and the denominator is the divisor. They were also aware of the relationship between decimals and fractions, and were very skillful in converting between the two number forms. Moreover, they were aware of how the models of division by fractions are connected to the meaning of multiplication with fractions and to whole number models of division.

As in the two previous chapters, the Chinese teachers did not regard the topic of this chapter as the key piece of the knowledge package in which it is included. The key piece in the package was the meaning of multiplication with fractions. The teachers regarded it as a "knot" that ties a cluster of concepts that support the understanding of the meaning of division by fractions. In the previous chapters we noted that the Chinese teachers tend to pay significant attention to the occasion when a concept is first introduced and tend to regard it as a key piece in a knowledge package. In addressing the key piece in the knowledge package of this chapter, they still adhered to this principle. However, since the mathematical topic discussed in this chapter is more advanced and complex, its stepping stone is not a single concept but a connection of several concepts.

One of the reasons why the U.S. teachers' understanding of the meaning of division of fractions was not built might be that their knowledge lacked connections and links. The understanding of most of the U.S. teachers was supported by only one idea—the partitive model of whole number division. Because other necessary concepts for understanding and their connections with the topic were missing, these teachers were not able to generate a conceptual representation of the meaning of division by fractions.

Relationship Between Teachers' Subject Matter Knowledge and Their Representations

Generating representations for a mathematical concept is a common teaching task. Most of the U.S. teachers tended to represent the meaning of division by fractions with a real-world example. The topics that the Chinese teachers used, however, were broader and less connected with students' lives. Doubtless connecting school mathematics learning with students' out-of-school lives may help them make more sense of mathematics. However, the "real world" cannot produce mathematical content by itself. Without a solid knowledge of what to represent, no matter how rich one's knowledge of students' lives, no matter how much one is motivated to connect mathematics with students' lives, one still cannot produce a conceptually correct representation.

SUMMARY

This chapter investigated teachers' subject matter knowledge of two aspects of the same topic—division by fractions. Teachers were asked to calculate $1\frac{3}{4} \div \frac{1}{2}$ and to illustrate the meaning of the operation, an aspect of subject matter knowledge not approached in previous chapters. The U.S. teachers' knowledge of division by fractions was obviously weaker than their knowledge of the two previous topics. Although 43% of the U.S. teachers succeeded in correctly calculating a complete answer, none showed an understanding of the rationale underlying their calculations. Only Tr. Belle, an experienced teacher, succeeded in generating a representation that correctly illustrated the meaning of division by fractions.

The Chinese teachers' performance on the task for this chapter was not noticeably different from that on the previous tasks. All of their calculations were correct and a few teachers went a step further to discuss the rationale underlying the algorithm. Most of the teachers generated at least one correct and appropriate representation. Their ability to generate representations that used a rich variety of subjects and different models of division by fractions seemed to be based on their solid knowledge of the topic. On the other hand, the U.S. teachers, who were unable to represent the operation, did not correctly explain its meaning. This suggests that in order to have a pedagogically powerful representation for a topic, a teacher should first have a comprehensive understanding of it.