MATH 149 **Spring 2010**

Study Guide and Sample Problems for Test 1

Note: the actual test will consist of five or six problems, some of which will be computational, some will ask for a brief explanation, and some will require a rigorous detailed proof. Some of the problems will be very similar to homework problems and/or those discussed in class, but some will be different. So make sure that you understand well all the concepts discussed, know precise definitions and basic properties, rather than memorize how to solve specific problems.

1. Groups, abelian groups, rings, commutative rings, fields.
   - Give the definition of a group; abelian group.
   - Give an example of a ring that is not a field.
   - Which of the following sets with the usual addition and multiplication are fields: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}^+$ (the set of positive real numbers), $\mathbb{R}$? For those the are not fields, say which axioms do not hold.
2. Equivalence relations.
   - Define a relation on $\mathbb{N} \times N$ by $(a, b) \sim (c, d)$ if $a + d = c + b$. Show that this relation is an equivalence relation. Find the equivalence class of $(3, 1)$. Describe all equivalence classes.
   - Define a relation on the set of Cauchy sequences of rational numbers by $\{a_i\} \sim \{b_i\}$ if $\lim_{i\to\infty}(a_i - b_i) = 0$. Show that this relation is an equivalence relation. What do all the sequences in each equivalence class have in common?
   - Define a relation on R by $a \sim b$ if $a - b \in \mathbb{Q}$. Show that this relation is an equivalence relation. What is the equivalence class of 1? Are $\sqrt{2}$ and $\sqrt{3}$ in the same equivalence relation? (Explain!)
   - Show that congruence modulo a natural number n is an equivalence relation.
3. Divisibility and congruences.
   - State and prove the divisibility test for 3.
   - Prove that if $a|b$ and $a|c$, then $a|(b + c)$.
   - Prove that for any $i \leq j$, 2^i divides 10^j.
   - Prove that $a \equiv b \pmod{10}$ if and only if a and b have the same units digit.
4. Rational and irrational numbers.
   - Prove that the quotient of two nonzero rational numbers is rational.
   - Prove that the following statement is false: "the quotient of two nonzero irrational numbers is irrational".

- Prove that $\sqrt{5}$ is irrational.
- Convert 12.3(45) into a fraction.
- Explain why a decimal represents a rational number if and only if it is either terminating or periodic.

5. Real numbers.
   - Give the definition of a convergent sequence.
   - Give the definition of a Cauchy sequence. What important property do all Cauchy sequences have?
   - Explain why the infinite decimal 0.(9) represents the same number as 1.
   - Show that $\sqrt{2} \notin \mathbb{Q}(\sqrt{3})$.
   - Find the multiplicative inverse of the element $2 + 5\sqrt{3}$ in $\mathbb{Q}(\sqrt{3})$.
6. Complex numbers.
   - Find the additive and multiplicative inverses of the element $4 + 3i$ in $\mathbb{C}$.
   - Compute $(\sqrt{3} + i)^9$.
   - Find at least one complex solution to $x^4 = -1$.
   - Give the definition of a convergent sequence.
   - Find the sum of the infinite series $1 + \frac{1}{2}i + \frac{1}{4} + \frac{1}{8}i + \frac{1}{16} + \frac{1}{32}i + \ldots$.