MATH 149 **Spring 2009**

Answers to Sample Problems for Test 2

1. Binomial theorem and probability
   - Explain why there are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ways to choose k objects out of n.
     There are n ways to choose the first object, $n-1$ ways to choose the second object, and so on. Thus there are $n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$ ways to choose k objects and put them in a certain order. For a similar reason, there are $k!$ ways to order k objects, however, we do not care about the order, so there are $\frac{n!}{k!(n-k)!}$ ways to choose k objects.
   - Expand: $(x+y)^n$.
     $(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \dots \binom{n}{k}x^{n-k}y^k + \dots \binom{n}{n-1}xy^{n-1} + y^n$
   - Find the first three terms in the expansions of $(2x+1)^5$, $(x-2y)^{10}$.
     $(2x+1)^5 = (2x)^5 + 5(2x)^4 + 10(2x)^3 + \dots = 32x^5 + 80x^4 + 80x^3 + \dots$
     $(x-2y)^{10} = x^{10} + 10x^9(-2y) + 45x^8(-2y)^2 + \dots = x^{10} - 20x^9y + 180x^8y^2 + \dots$
   - Prove that in Pascal's tringle, each number is equal to the sum of the two numbers above it.
     This is equivalent to proving that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$: done in class (rewrite each side in terms of factorials and prove the identity).
   - How many ways are there to choose 3 cards from a deck of 52 cards? How many ways are there to choose 3 cards from the 12 "face" cards (J, Q, K)? If three cards are chosen randomly for a deck of 52 cards, what is the probability that all three are face cards?
     There are $\frac{52\cdot 51\cdot 50}{6} = 22100$ ways to choose 3 cards from a deck of 52 cards;
     $\frac{12\cdot 11\cdot 10}{6} = 220$ ways to choose 3 cards from the 12 "face" cards;
     the probability is $\frac{220}{22100} = \frac{11}{1105}$.
2. Polynomials
   discussed in class
3. Functions
   discussed in class
4. Equations and inequalities
   - Derive or prove the quadratic formula.
     I.e. derive using completing the square or prove by multiplying out $(x-r_1)(x-r_2)$ and showing that the equation $(x-r_1)(x-r_2)=0$ is equivalent to $ax^2+bx+c=0$
   - What does it mean to solve an equation or an inequality?
     It means to find all roots (all values that satisfy the equation or inequality).

- Is guess and check a good method for solving equations? Explain, provide examples to support your arguments.
  discussed in class
- Solve: $\sqrt{(x-2)(x-7)} = 5\sqrt{2}$.
  $(x-2)(x-7) = 50$
  $x^2 - 9x - 36 = 0$
  $(x-12)(x+3) = 0$
  $x = 12,\ x = -3$
- Solve: $\log_{2x-3} 9 = 2$.
  $(2x-3)^2 = 9$
  $2x - 3 = 3$
  $x = 3$
- Solve: $\tan^3 x - \tan^2 x - 3\tan x + 3 = 0$.
  let $y = \tan x$
  then $y^3 - y^2 - 3y + 3 = 0$
  $(y^2 - 3)(y - 1) = 0$
  $y = 1,\ y = \pm\sqrt{3}$
  $\tan x = 1,\ \tan x = \pm\sqrt{3}$
  $x = \pi/4 + \pi n,\ x = \pm\pi/3 + \pi n,\ n \in \mathbb{Z}$
- Solve: $x^2 < 15$; $x^2 > 15$; $x^2 \le 15$; $x^2 \ge 15$; $12 < x^2 < 15$.
  $(-\sqrt{15}, \sqrt{15})$; $(-\infty, -\sqrt{15}) \cup (\sqrt{15}, \infty)$; $[-\sqrt{15}, \sqrt{15}]$; $(-\infty, -\sqrt{15}] \cup [\sqrt{15}, \infty)$; $(-\sqrt{15}, -\sqrt{12}) \cup (\sqrt{12}, \sqrt{15})$
- Solve: $x^{x+1} = 8$
  $x = 2$
- How many integer solutions does the following system of inequalities have?

  $$x < 2y, \quad x + y > 3, \quad y < 4$$

  7 (sketch the inequalities)

5. Proportional reasoning, ratios, and rates

- If $5x = 6y$, what is the ratio of x to y?
  $6 : 5$
- The distance fallen by a freely falling body is proportional to the square of the time it has been falling. If an object falls 144 feet in 3 seconds, how far will it fall in 6 seconds?
  $d = kt^2$
  $144 = k \cdot 9$
  $k = 16$
  so $d = 16t^2$
  $d = 16 \cdot 6^2 = 576$
- If Misha can eat a large box of oatmeal in two months and Katya can eat the same box in one month, in how much time will they eat one box together?

In two months Misha will eat one box and Katy will eat two boxes, so together they will eat 3 boxes. Therefore they will eat one box in 2/3 of a month.

- Luke can solve 5 problems in 10 minutes. How many problems can he solve in 30 minutes? Five hours? One week?
  discussed in class