2014
LEAP FROG RELAY GRADES 11-12
PART I SOLUTIONS

No calculators allowed
Correct Answer $= 4$, **Incorrect Answer** $= -1$, **Blank** $= 0$

(1) Let r be the remainder of $1 + 2^2 + 3^3 + 4^4 + 5^5 + 6^6 + 7^7 + 8^8 + 9^9 + 10^{10}$ when divided by 3. Let s be the sum of the last digits of each of the terms o

(a) 47 (b) 49

(c) 45 (d) 42

(e) None of the above

Solution: (b) We first eliminate the multiples of 3 from the given sum, as they will not contribute to the remainder of the sum, and get $1 + 2^2$ In order to find s we find, by hand, the last digits of the first few terms in the sum. They are: $1, 4, 7, 6, 5, 6$ (the last two because every power of 5 ends in$s = 1 + 4 + 7 + 6 + 5 + 6 + 3 + 6 + 9 + 0 = 47$.Thus $r + s = 2 + 47 = 49$.

Let

$$S = \sqrt{1} + \sqrt{1+2^3} + \sqrt{1+2^3+3^3} + \cdots + \sqrt{1+2^3+3^3+\cdots+2014^3}$$

Then,

(a) $S = \binom{2016}{3}$

(b) $S = \binom{2014}{3}$

(c) $S = \binom{2016}{6}$

(d) $S = \binom{2013}{6}$

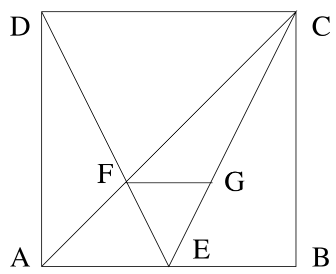
(e) None of the above

Here, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Solution: (a) Use $\displaystyle\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \left(\frac{n(n+1)}{2}\right)^2$ to get

$$\begin{aligned}
S = \sum_{n=1}^{2014} \frac{n(n+1)}{2} &= \frac{1}{2}\sum_{n=1}^{2014}(n^2+n) \\
&= \frac{1}{2}\left(\sum_{n=1}^{2014} n^2 + \sum_{n=1}^{2014} n\right) \\
&= \frac{1}{2}\left(\frac{2014(2015)(2\cdot 2014+1)}{6} + \frac{2014(2015)}{2}\right) \\
&= \frac{2014(2015)}{4}\left(\frac{2\cdot 2014+1}{3} + 1\right) \\
&= \frac{2014(2015)}{4}\left(\frac{2\cdot 2014+4}{3}\right) \\
&= \frac{2014(2015)}{2}\left(\frac{2016}{3}\right) \\
&= \binom{2016}{3}
\end{aligned}$$

The square $ABCD$ has sides of length 2. Point E is the midpoint of edge AB. Point F is the intersection of lines AC and DE. Line FG is parallel to line AB. The area of $\triangle EFG$ is:



(a) $\frac{2}{3}$

(b) $\frac{1}{3}$

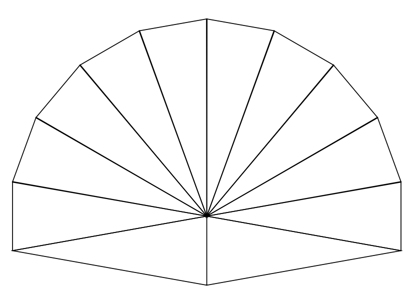
(c) $\frac{2}{9}$

(d) $\frac{4}{9}$

(e) None of the above

Solution: (c) First notice that AE has length 1, and thus $\triangle ACE$ has base 1 and height 2, hence its area is 1. Next we use that $\triangle AEF \sim \triangle CDF$, and that AE has half the length of CD, to get that AF must have half the length of CF. Also because FG is parallel to AE, we must have that EG is half the length of CG.
Now consider $\triangle ACE$ with AC as its base. By comparing the ratio of AF and CF, we conclude that the area of $\triangle CFE$ is 2/3 of the area of $\triangle ACE$. Hence the area of $\triangle CFE$ is 2/3.
Finally, consider $\triangle CFE$ with EC as its base. By comparing the ratio of EG and CG, we conclude that the area of $\triangle EFG$ is 1/3 of the area of $\triangle CFE$. Therefore, the area of $\triangle EFG$ is $(1/3)(2/3) = 2/9$.

An isosceles triangle $\triangle ABC$ has equal angles $B = C$. Twelve copies of $\triangle ABC$ are arranged around a common vertex without gaps or overlaps as shown. (The common vertex is surrounded by 10 angles equal to A and 2 angles equal to B.) Find the measure of A in degrees.



(a) $15°$ (b) $25°$

(c) $18°$ (d) $20°$

(e) None of the above

Solution: (d) Recall that the sum of the angles in a triangle is $180°$, and that the angle measure around a point is $360°$. This tells us that $\angle A + 2\angle B = 180°$ and that $10\angle A + 2\angle B = 360°$. These two equations combine to imply that $9\angle A = 180°$, therefore $\angle A = 20°$.

Let $S = \{1, 4, 9, 16, 25, ...\}$ be the set of squares of positive integers. Let $t \in S$ be such that $t - 76 \in S$. What is $76t$?

(a) $30,400$ (b) $27,436$

(c) $24,624$ (d) $33,516$

(e) None of the above

Solution: (a) We let $t = x^2$, and so we want to solve $x^2 - 76 = y^2$. We rewrite this equation as $76 = x^2 - y^2 = (x - y)(x + y)$. We note that $(x - y)$ and $(x + y)$ have the same parity (they are either both even or both odd), and since their product is even, it follows that $(x - y)$ and $(x + y)$ must be both even.
Since 76 factors as $1 \cdot 76$, $2 \cdot 38$, or $4 \cdot 19$ then the only factorization that could be matched to $(x - y)(x + y)$ is $2 \cdot 38$, because of parity of the factors. It follows that $x = 20$ and $y = 18$. Hence $t = 400$, and thus $76t = 30,400$.

Given that $2^{60} = 1,152,921,504,606,846,976$, find the first four digits (reading left to right) of 2^{61} and 2^{59}, then add these 8 digits up to get:

(a) 35 (b) 30

(c) 32 (d) 28

(e) None of the above

Solution: (c) When multiplying 2^{60} by 2, the digit of '9' in the 5^{th} place (from the left) will cause a '1' to carry over to be added to the digit in the 4^{th} place from the left. So, the first four digits of 2^{61} are $2 \cdot (1152) + 1 = 2305$. Similarly, when dividing 2^{60} by 2, we start from the left, obtaining the digits 5764. It follows that the answer is $2 + 3 + 0 + 5 + 5 + 7 + 6 + 4 = 32$.

Given that a and f are integers between 0 and 9 such that $a^5 + 1 = f \cdot 1111$, find $a + f$.

(a) 15 (b) 13

(c) 12 (d) 10

(e) None of the above

Solution: (b) The only 5^{th} powers of the given digits that produce 4 digit numbers are $4^5 = 16 \cdot 64 = 1024$, $5^5 = 25 \cdot 125 = 3125$, and $6^5 = 36 \cdot 216 = 7776$. Note that $6^5 + 1 = 7777 = 7 \cdot 1111$, and that $1024 + 1 = 1025$ and $3125 + 1 = 3126$ are not divisible by 1111, we have that $a = 6$ and $f = 7$. Therefore, $a + f = 6 + 7 = 13$.

Three solutions of the equation $m!(m+1)! = n!$ are $(m, n) = (0, 0)$, $(m, n) = (0, 1)$, and $(m, n) = (1, 2)$. There is a unique fourth solution to this equation so that $0 \le n \le 10$ and $0 \le m \le 10$. For that solution, find $n - m$.

(a) 6 (b) 5

(c) 3 (d) 4

(e) None of the above

Solution: (d) We observe first that $m+1 < n$, since otherwise $(m+1)! \ge n!$. Now note that every prime number dividing the right-hand side of the equality must divide the left-hand side, as well. Thus, if there were a prime number p such that $m + 1 < p \le n$ then p would divide the right hand side but not the left hand side. In particular n cannot be prime. So, the only possible solutions to consider are

$$(m, n) = (2, 4), (4, 6), (6, 8), (6, 9), (7, 9), (6, 10), (7, 10), (8, 10)$$

We check these pairs and find out that $(m, n) = (6, 10)$ works. Therefore, the answer is $m - n = 10 - 6 = 4$.

Given that $2 + \sqrt{3}$ is one of the solutions of the equation

$$x^4 - 14x^3 + 54x^2 - 62x + 13 = 0$$

how many complex solutions does this equation have?

(a) 0 (b) 1

(c) 2 (d) 3

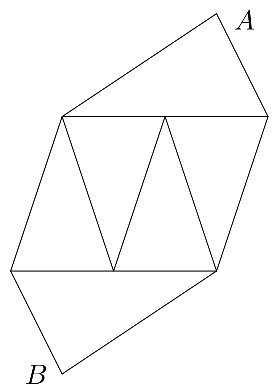
(e) 4

Solution: (a) Since we know $2+\sqrt{3}$ is one of the solutions of the equation (which has only integral coefficients) then so is $2-\sqrt{3}$. Then we have

$$[x-(2+\sqrt{3})][x-(2-\sqrt{3})] = (x-2)^2-(\sqrt{3})^2 = x^2-4x+1$$

We now divide $x^4-14x^3+54x^2-62x+13$ by x^2-4x+1 getting $x^2-10x+13$. The discriminant of this quadratic equation is $(10)^2-4\cdot 13 = 58$, which is positive, and thus it does not have any complex roots.

The adjacent figure has six non-overlapping congruent isosceles triangles. In each triangle the equal sides are 2 units and the base is 1 unit. Find the distance from A to B.



(a) $\sqrt{19}$ (b) $\sqrt{17}$

(c) $3\sqrt{2}$ (d) $2\sqrt{5}$

(e) None of the above

Solution: (a) We label the vertices of the parallelogram (composed of 4 triangles) in the figure as $OPQR$. If O is the origin $(0,0)$, then $|OP| = 2$, so $P = (2,0)$. Therefore the x-coordinate of R is $1/2$, and the altitude at R has length y where $y^2+(1/2)^2 = 4$, so $y = \sqrt{4-1/4} = \sqrt{15}/2$ and $R = (1/2, \sqrt{15}/2)$. Then M, the midpoint of OR has coordinates $(1/4, \sqrt{15}/4)$. By symmetry, since $\angle POB \cong \angle POR$, we see that point B is symmetric with M in segment OP, so $B = (1/4, -\sqrt{15}/4)$. By a similar argument, the coordinates of point A are $(9/4, 3\sqrt{15}/4)$. This gives us the distance $|AB|$ as

$$|AB|^2 = \left(\frac{9}{4}-\frac{1}{4}\right)^2 + \left(\frac{3\sqrt{15}}{4}+\frac{\sqrt{15}}{4}\right)^2 = 4+15 = 19.$$