2014
LEAP FROG RELAY GRADES 11-12
PART II SOLUTIONS

No calculators allowed
Correct Answer $= 4$**, Incorrect Answer** $= -1$**, Blank** $= 0$

(1) Let $f(x) = |3x - 2|$. Find the sum of *all* real solutions, x, to the equation $f(f(x)) = 2$.

(a) 2
(b) $\dfrac{14}{9}$
(c) $\dfrac{16}{3}$
(d) 0
(e) None of the above

Solution: (a) The equation $f(f(x)) = 2$ can be re-written as $|3|3x-2|-2| = 2$, which is equivalent to $3|3x-2| - 2 = \pm 2$, or to $3|3x-2| = 2 \pm 2$. So we have:

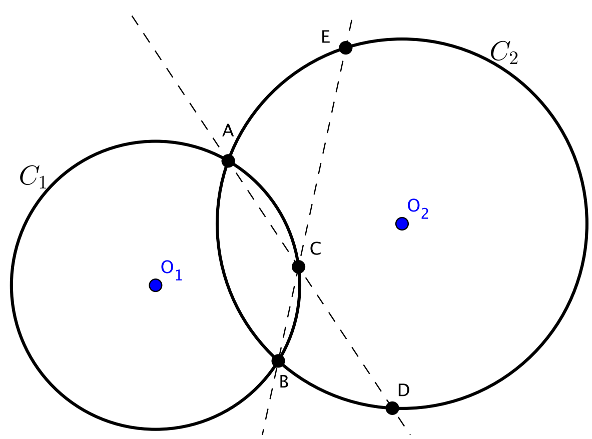
$$|3x - 2| = \frac{4}{3} \qquad \text{or} \qquad |3x - 2| = 0$$

The above equations give us:

$$3x - 2 = \pm\frac{4}{3} \qquad \text{or} \qquad 3x - 2 = 0$$

So, the solutions to the given equation are $\dfrac{10}{9}$, $\dfrac{2}{9}$ and $\dfrac{2}{3}$, whose sum is 2.

(2) Circles C_1 and C_2, centered at O_1 and O_2 respectively, intersect in A and B. Point C is on circle C_1, and AC, BC meet circle C_2 in D and E, respectively. Find the measure of the angle between $\overleftrightarrow{O_1C}$ and $\overleftrightarrow{DE}$.

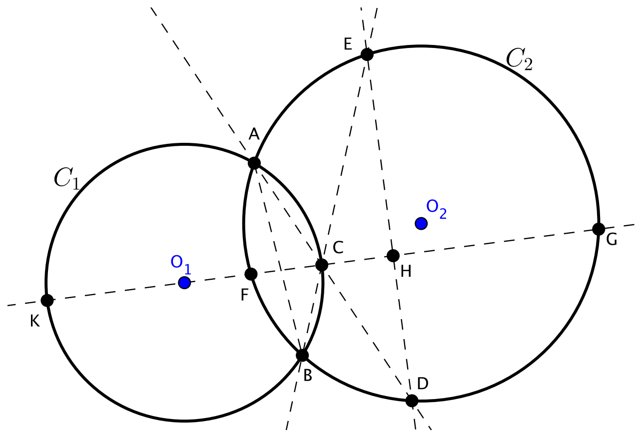


(a) $92°$ (b) $95°$

(c) $90°$ (d) $88°$

(e) None of the above

Solution: (c) Let H be the intersection of O_1C and DE, and assume O_1C meet C_2 in F and G, and C_1 in K, as shown in the picture below.



Then

$$\angle CDH \cong \angle ADE \cong \angle ABE \cong \angle ABC$$

which is half of the arc AC. Also,

$$\angle DCH \cong \angle ACK$$

which is half of the arc AK.
It follows that $\angle CDH + \angle DCH$ is half of the arc CK, which means that

$$\angle CDH + \angle DCH = 90°.$$

But then (from $\triangle CDH$) we get

$$\angle CHD = 180° - (\angle CDH + \angle DCH) = 90°$$

(3) Find the sum of all of the real numbers x which satisfy

$$\sin x + \cos x = \sqrt{\frac{2+\sqrt{3}}{2}}$$

with $0 < x < \pi/2$.

(a) $\pi/2$ (b) $\pi/6$

(c) $2\pi/3$ (d) $\pi/6$

(e) None of the above

Solution: (a) We square both sides of the equation given, and use that $\sin^2 x + \cos^2 x = 1$, to get

$$1 + 2\sin x \cos x = \frac{2+\sqrt{3}}{2}$$

which can be re-written as

$$\sin(2x) = \frac{\sqrt{3}}{2}$$

It follows that for $0 < 2x < \pi$ we get $2x = \pi/3, 2\pi/3$, and thus for $0 < x < \pi/2$ we get $x = \pi/6, \pi/3$, whose sum equals $\pi/2$.

(4) Let $a > 1$ and $b > 1$ be real numbers such that

$$\log_{10}(a+b) = \log_{10} a + \log_{10} b$$

What can you say about the value of

$$\log_{10}(a-1) + \log_{10}(b-1)?$$

(a) 2 (b) 3

(c) 1 (d) 0

(e) None of the above

Solution: (d) Since $\log_{10}(ab) = \log_{10} a + \log_{10} b$, the equation $\log_{10}(a+b) = \log_{10} a + \log_{10} b$ really means $\log_{10}(a+b) = \log_{10}(ab)$, which is equivalent to $\log_{10}(a+b) - \log_{10}(ab) = 0$. But this can be re-written as

$$0 = \log_{10}(a+b) - \log_{10}(ab) = \log_{10}\left(\frac{a+b}{ab}\right)$$

which implies that $\frac{a+b}{ab} = 1$, and thus $a + b = ab$. Using this, we compute

$$\log_{10}(a-1)+\log_{10}(b-1) = \log_{10}(a-1)(b-1) = \log_{10}(ab-a-b+1) = \log_{10} 1 = 0$$

(5) A working crew of x men work x hours a day for x days to dig a tunnel of length x yards. A second crew of y men work y hours a day for y days. What length (in yards) of the continuation of the tunnel would you expect them to dig? You may assume all men work at the same rate.

(a) y (b) x^2/y^3

(c) y^3/x^2 (d) y^2/x^2

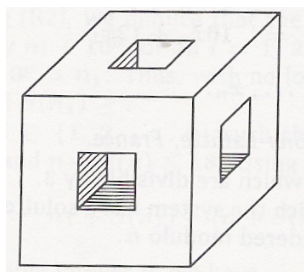
(e) None of the above

Solution: (c) Since the first crew works x days to get x yards of tunnel, then they dig one yard per day. So we need x men working x hours to get 1 yard dug. That is one man would need to work x^2 hours to dig one yard of tunnel.
Let z be the number of yards dug by the second crew. An argument similar to the one used for the first crew says that a man in the second crew would need to work y^2 hours to dig z/y yards of tunnel.
Since all men work at the same rate we get

$$\frac{x^2}{1} = \frac{y^2}{\frac{z}{y}}$$

which forces $z = \dfrac{y^3}{x^2}$.

(6) In a cube of side 3 in, in the center of three different (and not opposite) faces we bore a square perforation of side 1 inch that goes across the cube as far as the opposite face. We thus obtain the following figure:



Determine the surface area of the resulting solid

(a) 72 in^2
(b) 70 in^2
(c) 68 in^2
(d) 74 in^2
(e) None of the above

Solution: (a) Each face of the cube has area 3^2 in^2, but with the perforation, each side now has area $3^2 - 1 = 8$ in^2. On each of the six faces there is a perforation with depth 1 in that is 'attached' to that face and to no other. Each one of these holes contributes to the surface area $4 \cdot 1 = 4$ in^2. Besides these areas found, there are no others. So, the total surface is

$$6 \cdot 8 + 6 \cdot 4 = 48 + 24 = 72 \; in^2$$

(7) Simplify

$$\sqrt{\frac{8^{10} + 4^{10}}{8^4 + 4^{11}}}$$

(a) 64 (b) 32

(c) 16 (d) 8

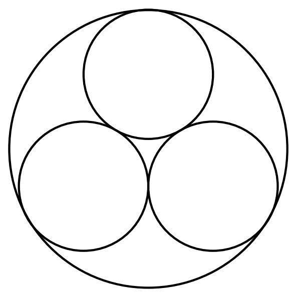
(e) None of the above

Solution: (c) We use that $8^n = 2^n \cdot 4^n$ to get

$$\sqrt{\frac{8^{10}+4^{10}}{8^4+4^{11}}} = \sqrt{\frac{4^{10}(2^{10}+1)}{4^4(2^4+4^7)}} = \frac{4^5}{4^2}\sqrt{\frac{2^{10}+1}{2^4+4^7}} = 4^3\sqrt{\frac{4^5+1}{4^2+4^7}} = 4^3\sqrt{\frac{4^5+1}{4^2(1+4^5)}} = \frac{4^3}{4}$$

which is equal to 16.

(8) Given three distinct unit circles (i.e. circles of radius 1), each of which is tangent to the other two, find the radius of the circle which is tangent to all three circles and contains them.



(a) $\dfrac{2\sqrt{3}+1}{3}$ (b) $\dfrac{2\sqrt{3}}{3}$

(c) $2\sqrt{3}$ (d) $\sqrt{3}+1$

(e) None of the above

Solution: (e) Joining the centers of the three unit circles we get an equilateral triangle $\triangle ABC$ with side 2. The incenter/circumcenter of this triangle is the center of the circle tangent to all three circles. Since in an equilateral triangle altitudes, angle bisectors, and side bisectors are all the same, we get that the altitude of $\triangle ABC$ is $\sqrt{3}$ (Pythagorean theorem). Also, since the altitudes are concurrent we get a right triangle with sides x, $\sqrt{3}-x$ and 1, where x is the distance from a vertex to the incenter. Hence, using the Pythagorean theorem again we get that $x = \frac{2\sqrt{3}}{3}$. Since the radius of the large circle is 1 (radius of a small circle) plus the distance from a vertex to the incenter, it is equal to $1+\dfrac{2\sqrt{3}}{3} = \dfrac{2\sqrt{3}+3}{3}$.

(9) Compute the integer k, $k > 2$, for which

$$\log_{10}[(k-2)!] + \log_{10}[(k-1)!] + 2 = 2\log_{10}(k!).$$

(a) $k = 4$

(b) $k = 5$

(c) $k = 7$

(d) $k = 6$

(e) None of the above

Solution: (b) Since $(k-1)! = (k-1)(k-2)!$ and $k! = k(k-1)(k-2)!$ then $\log_{10}[(k-2)!] + \log_{10}[(k-1)!] + 2 = 2\log_{10}(k!)$ can be re-written as

$$\log_{10}[(k-2)!]+\log_{10}[(k-2)!]+\log_{10}(k-1)+2 = 2\left(\log_{10}[(k-2)!] + \log_{10}(k-1) + \log_{10} k\right)$$

which simplifies to

$$2 = \log_{10}(k-1) + 2\log_{10} k$$

It follows that $\log_{10}[(k-1)k^2] = 2$, and so $(k-1)k^2 = 10^2$. Since k is an integer, we need $k-1$ to be a square. A quick search shows that $k = 5$.

(10) The smallest prime number that divides $2^{111} + 3^{111}$ is

(a) 23

(b) $2^{111} + 1$

(c) 17

(d) $3^{111} + 1$

(e) None of the above

Solution: (e) The unit digits of the numbers 2^n and 3^n cycle in the repeated patters $2, 4, 8, 6$ and $3, 9, 7, 1$ respectively. Adding the units digit, to get the units digit of $2^n + 3^n$, gives the repeating pattern $5, 3, 5, 7$. So we see the units digit of $2^{111} + 3^{111}$ is 5 because 111 is odd. This means that 5 is a prime dividing $2^{111} + 3^{111}$. Since neither 2 nor 3 are factors, 5 is the smallest prime factor of $2^{111} + 3^{111}$.