2013
Leap Frog Relay Grades 11-12
Part II (problems 11-20)

No calculators allowed
Correct Answer $= 4$ **points**
Incorrect Answer $= -1$ **point**
Blank $= 0$ **points**

11. Let C be a circle that intersects each of the circles $(x+2)^2 + y^2 = 2^2$, $(x-4)^2 + (y-2)^2 = 2^2$, and $(x-4)^2 + (y+2)^2 = 2^2$ in exactly one point, and does not contain any of these circles inside it. If the radius r of C is of the form $r = \frac{p}{q}$ (in lowest terms), what is $p+q$?

    (a) 5 (b) 7

    (c) 9 (d) 11

    (e) 13

12. You are in a large room with 50 ceiling lights (numbered from 1 to 50) that are changed from on to off or off to on by pulling a cord hanging from each light. Initially, all the lights are off. You begin by pulling the cord on every light (now they are all on). Then you pull the cord on light $2, 4, 6, \ldots, 48, 50$. After you finish that, you pull the cord on light $3, 6, 9, \ldots, 45, 48$. You repeat this with every fourth light, every fifth light, etc. until you pull the cord for every 50th light (only number 50, of course). How many lights are on at the end?

    (a) 1 (b) 2

    (c) 3 (d) 5

    (e) 7

13. Find all solutions for θ on $[0, \pi)$:

$$\log_{\sin\theta}\left(\log_{49}\sqrt{7}\right) = 2$$

(a) $\frac{\pi}{4}$ and $\frac{3\pi}{4}$

(b) $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

(c) $\pm 1/2$

(d) $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$

(e) $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$

14. Find the imaginary parts of the roots of

$$iz^2 + (2+i)z + 1$$

(a) $\frac{-1 \pm \sqrt{3}}{2}$

(b) $\frac{-2 \pm \sqrt{3}}{2}$

(c) $\frac{1 \pm \sqrt{3}}{2}$

(d) $\frac{2 \pm \sqrt{3}}{2}$

(e) None of the above

15. Consider the following equation where m and n are real numbers:

$$(x^2 - 2x + m)(x^2 - 2x + n) = 0$$

Suppose the four roots of the equation form an arithmetic sequence with the first (and smallest) term being $1/4$. What is the value of $|m - n|$?

(a) $\frac{3}{8}$

(b) $\frac{1}{2}$

(c) $\frac{5}{8}$

(d) $\frac{3}{4}$

(e) 1

16. When an orchestra plays a national anthem, its musicians are ordered in a square. When the orchestra plays any other song, the musicians are ordered in a rectangle such that the number of rows increases by five. What is the number m of musicians in the orchestra?

(a) $333 \leq m < 444$

(b) $222 \leq m < 333$

(c) $111 \leq m < 222$

(d) $50 \leq m < 111$

(e) $m < 50$

17. Let $\triangle ABC$ be an equilateral triangle with an inscribed circle of radius 1. Find the length of AB.

(a) $\sqrt{2}$ (b) $2\sqrt{2}$

(c) $\sqrt{3}$ (d) $2\sqrt{3}$

(e) $3\sqrt{3}$

18. The four numbers $a < b < c < d$ can be paired in six different ways. If each pair has a different sum, and if the four smallest sums are 1, 2, 3, and 4, what are all possible values of d?

(a) 4 (b) $\frac{7}{2}$ and 4

(c) $\frac{7}{2}$ (d) $\frac{3}{2}$ and 4

(e) $\frac{3}{2}$ and $\frac{7}{2}$

19. In the English alphabet of capital letters, there are 15 'stick' letters which contain no curved lines, and 11 'round' letters which contain at least some curved segment. How many different 3-letter sequences can be made of two different stick letters and one curved letter?

Stick: $A\ E\ F\ H\ I\ K\ L\ M\ N\ T\ V\ W\ X\ Y\ Z$
Round: $B\ C\ D\ G\ J\ O\ P\ Q\ R\ S\ U$

(a) 2310 (b) 4620

(c) 6930 (d) 13860

(e) None of the above

20. How many integers of the form n^4+4, where n is a non-negative integer, are prime? *Hint:* Complete the square.

(a) 0 (b) 1

(c) 2 (d) 3

(e) 4