2013
Leap Frog Relay Grades 11-12
Part II

No calculators allowed
Correct Answer $= 4$**, Incorrect Answer** $= -1$**, Blank** $= 0$

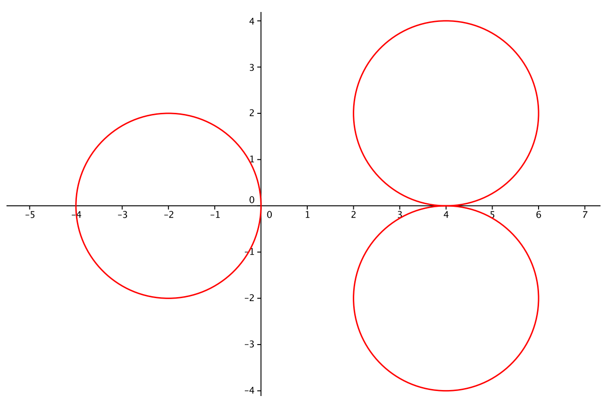
1. Let C be a circle that intersects each of the circles $(x+2)^2 + y^2 = 2^2$, $(x-4)^2 + (y-2)^2 = 2^2$, and $(x-4)^2 + (y+2)^2 = 2^2$ in exactly one point, and does not contain any of these circles inside it. If the radius r of C is of the form $r = \dfrac{p}{q}$ (in lowest terms), what is $p+q$?

   (a) 5 (b) 7

   (c) 9 (d) 11

   (e) 13

   Solution (e): A picture shows



and thus, by symmetry, the center of C must be on the x-axis, but then C being tangent to the circle on the left forces the center of C to be at $(r, 0)$, where r is the radius of C. We next look at the distances between this point and the centers of the three circles. We get $(r+2)^2 = (r-4)^2 + (0-2)^2$. Solving for r, we get $r = \dfrac{8}{5}$.

2. You are in a large room with 50 ceiling lights (numbered from 1 to 50) that are changed from on to off or off to on by pulling a cord hanging from each light. Initially, all the lights are off. You begin by pulling the cord on every light (now they are all on). Then you pull the cord on light $2, 4, 6, \ldots, 48, 50$. After you finish that, you pull the cord on light $3, 6, 9, \ldots, 45, 48$. You repeat this with every fourth light, every fifth light, etc. until you pull the cord for every 50th light (only number 50, of course). How many lights are on at the end?

   (a) 1 (b) 2

   (c) 3 (d) 5

   (e) 7

   Solution (e): The main idea is that a cord must be pulled an odd number of times for the light to be on. So, if we knew the number of divisors of a given number $1 \leq n \leq 50$ then we would know whether or not it ends up being on. So, let $n = p_1^{a_1} \cdots p_k^{a_k}$ be the prime factorization of n. We know that the number of positive divisors of n is given by $(a_1+1)(a_2+1)(a_k+1)$. In order for this number to be odd we need every factor to be odd, and thus every a_i must be even. In other words, we need n to be a perfect square. It follows that lights $1, 4, 9, 16, 25, 36$, and 49 are on at the end.

3. Find all solutions for θ on $[0, \pi)$:

$$\log_{\sin\theta}\left(\log_{49}\sqrt{7}\right) = 2$$

   (a) $\dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$ (b) $\dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$

   (c) $\pm 1/2$ (d) $\dfrac{7\pi}{6}$ and $\dfrac{11\pi}{6}$

   (e) $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$ and $\dfrac{11\pi}{6}$

   Solution (b): Since $\log_{49}\sqrt{7} = \log_{49}\left(7^{1/2}\right) = \log_{49}\left(49^{1/4}\right)$ then $\log_{49}\sqrt{7} = \dfrac{1}{4}$. So, the equation to solve is $\log_{\sin\theta}\left(\dfrac{1}{4}\right) = 2$, which is equivalent to $\sin^2\theta = \dfrac{1}{4}$. Hence, $\sin\theta = \pm\dfrac{1}{2}$, and thus $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$. Only the first two live in $[0, \pi)$.

4. Find the imaginary parts of the roots of

$$iz^2 + (2+i)z + 1$$

(a) $\frac{-1 \pm \sqrt{3}}{2}$ (b) $\frac{-2 \pm \sqrt{3}}{2}$

(c) $\frac{1 \pm \sqrt{3}}{2}$ (d) $\frac{2 \pm \sqrt{3}}{2}$

(e) None of the above

Solution (d): We use the quadratic formula to get that the roots are given by

$$\frac{-(2+i) \pm \sqrt{(2+i)^2 - 4i}}{2i} = \frac{-(2+i) \pm \sqrt{3}}{2i} = \frac{-2 \pm \sqrt{3}}{2i} - \frac{i}{2i}$$

which after multiplying top and bottom by i yields

$$\frac{2 \pm \sqrt{3}}{2} i - \frac{1}{2}$$

5. Consider the following equation where m and n are real numbers:

$$(x^2 - 2x + m)(x^2 - 2x + n) = 0$$

Suppose the four roots of the equation form an arithmetic sequence with the first (and smallest) term being $1/4$. What is the value of $|m - n|$?

(a) $\frac{3}{8}$ (b) $\frac{1}{2}$

(c) $\frac{5}{8}$ (d) $\frac{3}{4}$

(e) 1

Solution (b): The four roots are

$$\frac{1}{4} \qquad \frac{1}{4} + k \qquad \frac{1}{4} + 2k \qquad \frac{1}{4} + 3k$$

Since the coefficient on x is 2 in both factors, one of the two quadratic expressions factors as $\left(x - \frac{1}{4}\right)\left(x - \left(3k + \frac{1}{4}\right)\right)$ and the other as $\left(x - \left(k + \frac{1}{4}\right)\right)\left(x - \left(2k + \frac{1}{4}\right)\right)$. In addition, $3k + \frac{1}{2} = 2$ so $k = \frac{1}{2}$. If we let $m = \left(k + \frac{1}{4}\right)\left(2k + \frac{1}{4}\right)$ and $n = \frac{1}{4}\left(3k + \frac{1}{4}\right)$, we have $m - n = \frac{15}{16} - \frac{7}{16} = \frac{1}{2}$.

6. When an orchestra plays a national anthem, its musicians are ordered in a square. When the orchestra plays any other song, the musicians are ordered in a rectangle such that the number of rows increases by five. What is the number m of musicians in the orchestra?

(a) $333 \leq m < 444$ (b) $222 \leq m < 333$

(c) $111 \leq m < 222$ (d) $50 \leq m < 111$

(e) $m < 50$

Solution (a): Let the number of musicians in the orchestra be $m = x^2$. If we increase the number of rows by five, we must decrease the number of columns by, say, c. The positive integers x and c must satisfy the equation $x^2 = (x+5)(x-c)$, which is equivalent to $5c = (5-c)x$. Since the left hand side is positive then so is the right hand side. Thus $5 - c > 0$ and c is at most 4. Trying $c = 1, 2, 3, 4$ we see that only $c = 4$ yields an integer value of x. Thus, we must have $c = 4$, $x = 20$, and $x^2 = 400$.

7. Let $\triangle ABC$ be an equilateral triangle with an inscribed circle of radius 1. Find the length of AB.

(a) $\sqrt{2}$ (b) $2\sqrt{2}$

(c) $\sqrt{3}$ (d) $2\sqrt{3}$

(e) $3\sqrt{3}$

Solution (d): Let O be the center of the circle and M be the intersection point of the circle with AB. Then $\angle OAM = 30°$, with $OM = 1$, hence $OA = 2$. Thus $AM = \sqrt{3}$, and $AB = 2AM = 2\sqrt{3}$.

8. The four numbers $a < b < c < d$ can be paired in six different ways. If each pair has a different sum, and if the four smallest sums are 1, 2, 3, and 4, what are all possible values of d?

(a) 4 (b) $\frac{7}{2}$ and 4

(c) $\frac{7}{2}$ (d) $\frac{3}{2}$ and 4

(e) $\frac{3}{2}$ and $\frac{7}{2}$

Solution (b): The six possible sums are $a+b, a+c, a+d, b+c, b+d$, and $c+d$. Since $a < b < c < d$ then the smallest two sums are $a+b = 1$ and $a+c = 2$. We get that $b = 1-a$ and $c = 2-a$. Of the other four sums, the largest two are $b+d$ and $c+d$, so the remaining sums must be equal to 3 and 4. Since $c = b+1$ if $b+c = 3$ then $b = 1$, but this forces $a = 0$, and having $a+d = 4$ forces $d = 4$. If instead $b+c=4$ then $b=\frac{3}{2}$ and thus $a=-\frac{1}{2}$. But $a+d=3$, and thus $d=\frac{7}{2}$. So, the two possible values of d are $d=\frac{7}{2}$ and 4.

9. In the English alphabet of capital letters, there are 15 'stick' letters which contain no curved lines, and 11 'round' letters which contain at least some curved segment. How many different 3-letter sequences can be made of two different stick letters and one curved letter?

Stick:	$A\ E\ F\ H\ I\ K\ L\ M\ N\ T\ V\ W\ X\ Y\ Z$
Round:	$B\ C\ D\ G\ J\ O\ P\ Q\ R\ S\ U$

(a) 2310

(b) 4620

(c) 6930

(d) 13860

(e) None of the above

Solution (c): Without regard to order, there are $C(15,2)=105$ choices for the two stick letters and 11 choices for the round letter. Allowing for different orders, there are $3!=6$ ways to order the chosen letters, so the total number with order considered is $6\cdot 105\cdot 11=6930$.

10. How many integers of the form n^4+4, where n is a non-negative integer, are prime? *Hint:* Complete the square.

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

Solution (b): Notice that

$$\begin{aligned} n^4+4 &= (n^4+4n^2+4)-4n^2 \\ &= (n^2+2)^2-(2n)^2 \\ &= (n^2+2-2n)(n^2+2+2n) \end{aligned}$$

Clearly, for all non-negative integer n both factors are natural numbers larger than 1. So, when $n>1$, the number n^4+4 cannot be prime. The only choices left are $n=0,1$. It is easy to see that only $n=1$ works.