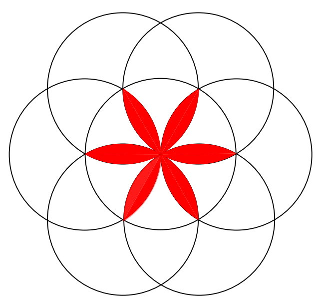
2013
Leap Frog Relay Grades 11-12
Part I

No calculators allowed
Correct Answer $= 4$**, Incorrect Answer** $= -1$**, Blank** $= 0$

1. Seven circles of radius 10 are arranged as in the figure. Note that the six outer circles all pass through the center of the inner circle, the inner circle passes through the center of each outer circle, and each outer circle passes through the center of the two outer circles it is adjacent to. The area of the shaded region is A. Which of the following is true about A?

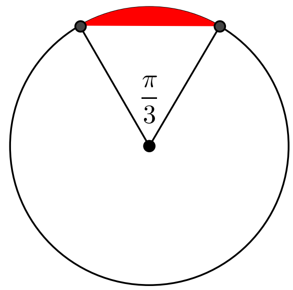


(a) $40 \leq A < 60$ (b) $60 \leq A < 80$

(c) $80 \leq A < 100$ (d) $100 \leq A < 120$

(e) $120 \leq A < 140$

Solution (d): We may assume the inner circle has center at $(0, 0)$ and for the outer circles, the centers are at angles $0, \pi, \pm\frac{\pi}{3}$, and $\pm\frac{2\pi}{3}$, exactly 10 units from $(0, 0)$. The colored portion is made up of twelve identical pieces as in the following figure



One gets these by starting with the wedge of a circle where the angle is $\frac{\pi}{3}$, connect the secant line and then remove the equilateral triangle (in this case with side length 10) formed. The remaining area is $100\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$. So, the total area is

$$12 \cdot 100\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = 100(2\pi - 3\sqrt{3})$$

To estimate the area, note that $17^2 = 289$, and $17.5^2 = 306.25$ ($18^2 = 324$), so $1.7 < \sqrt{3} < 1.75$. Thus the area is between 100 and 120.

2. From the list of all natural numbers $2, 3, \ldots, 999$, one deletes sublists of numbers nine times: At first, one deletes all even numbers, then all numbers divisible by 3, then all numbers divisible by 5, and so on, for the nine primes $2, 3, 5, 7, 11, 13, 17, 19, 23$. Find the sum of the composite numbers left in the remaining list.

   (a) 0 (b) 961

   (c) 3062 (d) 2701

   (e) 899

   Solution (d): The composite numbers left have all prime factors at least 29. There are three of them less than 1000: $29 \cdot 29 = 841$, $29 \cdot 31 = 899$ and $31 \cdot 31 = 961$, with the sum $841 + 899 + 961 = 2701$.

3. What are the solutions to

$$\tan\left(x + \frac{\pi}{6}\right) = \sqrt{3}$$

   where $0 \leq x \leq 2\pi$

   (a) $\frac{11\pi}{6}, \frac{5\pi}{6}$ (b) $\frac{5\pi}{3}, \frac{2\pi}{3}$

   (c) $\frac{\pi}{3}, \frac{4\pi}{3}$ (d) $\frac{\pi}{6}, \frac{7\pi}{6}$

   (e) None of the above

Solution (d): The solutions for $\tan\theta = \sqrt{3}$ with $0 \le \theta \le 2\pi$ are $\theta = \dfrac{\pi}{3}$ and $\theta = \pi + \dfrac{\pi}{3}$. Setting $x + \dfrac{\pi}{6}$ equal to these values yields

$$x + \frac{\pi}{6} = \frac{\pi}{3} \qquad\qquad x + \frac{\pi}{6} = \pi + \frac{\pi}{3}$$

which imply

$$x = \frac{\pi}{6} \qquad\qquad x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

4. Solve for x

$$\log_5\left(2x^2 - 3x + 1\right) - \log_5\left(x - 1\right) + \log_5 125 = 6$$

(a) 1

(b) $\dfrac{1}{2}$

(c) 5^3

(d) 63

(e) None of the above

Solution (d): $\log_5\left(2x^2 - 3x + 1\right) - \log_5\left(x - 1\right) + \log_5 125 = 6$ implies

$$\log_5\left(\frac{125(2x^2 - 3x + 1)}{x - 1}\right) = 6$$

It follows that

$$5^6 = \frac{125(2x^2 - 3x + 1)}{x - 1} = \frac{5^3(2x - 1)(x - 1)}{x - 1} = 5^3(2x - 1)$$

Hence, $5^3 = 2x - 1$, and thus $x = 63$.

5. Given a parabola with directrix $y = -2$ and focus $(2, 2)$. What are the value(s) for m for which the line $y = mx$ is tangent to the parabola?

(a) ± 4

(b) 0

(c) 2 and 4

(d) -2 and -4

(e) 0 and -8

Solution (e): The vertex of the parabola is midway between the directrix and the focus, and thus it is equal to $(2, 0)$. It is now easy to see that the equation of the parabola is $y = (x - 2)^2$. We plug a point (x, mx) of the line into the equation of the parabola to find intersection point(s). We get $mx = (x - 2)^2$, which implies $x^2 - (4 + m)x + 4 = 0$. We want the line to be tangent, so we want exactly one solution for that equation. Hence, we want the discriminant to be equal to zero. That is,

$$(4 + m)^2 - 4 \cdot 1 \cdot 4 = 0$$

which implies $m = 0$ or $m = -8$.

6. Consider the equation

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$$

where x represents a real number. How many solutions are there?

(a) Exactly one solution

(b) Exactly two solutions

(c) Exactly three solutions

(d) Exactly four solutions

(e) Infinitely many solutions

Solution (e): Make the substitution $z=\sqrt{x-1}$. Then the equation becomes $\sqrt{z^2+4-4z}+\sqrt{z^2+9-6z}=1$. Next note that $z^2-4z+4=(z-2)^2$ and $z^2-6z+9=(z-3)^2$. Thus the equation reduces to $|z-2|+|z-3|=1$. Note that every z in the interval $[2,3]$ is a solution to this equation, as if $2\le z\le 3$, then $|z-2|+|z-3|=z-2+3-z=1$. It follows that every x in $[5,10]$ is a solution of the original equation.

7. The points A, B, C, and D are the vertices of a unit square. How many squares (including $ABCD$ itself) in the same plane have two or more of these points as vertices?

(a) 13

(b) 12

(c) 9

(d) 5

(e) 4

Solution (a): There are squares of three sizes. There are 4 of area $\frac{1}{2}$, 5 of area 1, and 4 of area 2.

8. How many two-digit numbers are divisible by each of their digits?

(a) 9

(b) 10

(c) 11

(d) 13

(e) 14

Solution (e): We are looking for two-digit numbers $10a+b$ with the property that a and b both divide $10a+b$. Equivalently, we want that a divides b and b divides $10a$. If $a=1$ then b divides 10, thus $b\in\{1,2,5\}$ (three values). If $a=2$ then b is even and divides 20, thus $b\in\{2,4\}$ (two values). If $a=3$ then b is a multiple of 3 and divides 30, thus $b\in\{3,6\}$ (two values). If $a=4$ then $b\in\{4,8\}$ (two values). If a is at least 5, i.e., $a\in\{5,6,7,8,9\}$ then the only way for b be a multiple of a and a digit is $b=a$ (five values). Altogether we get fourteen solutions.

9. The Fibonacci numbers $1, 1, 2, 3, 5, 8, \ldots$ form a sequence where each term, after the first two, is the sum of the two previous terms. How many of the first 1000 terms are even?

(a) 222 (b) 333

(c) 499 (d) 500

(e) 501

Solution (b): The sequence of remainders, when we divide by 2, is $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$, i.e., a periodic sequence that repeats itself after each third entry. Hence the number of even entries among the first thousand Fibonacci numbers is the same as the number of positive integers up to 1000 that are multiples of 3.

10. One sphere is inscribed in a cube, while the cube is also inscribed in another sphere. Find the ratio of the volumes of the larger sphere to the smaller sphere.

(a) $\sqrt{3}$ (b) $2\sqrt{3}$

(c) $3\sqrt{3}$ (d) $3\sqrt{2}$

(e) None of the above

Solution (c): The inscribed sphere has a radius r and $V = \frac{4}{3}\pi r^3$, while the circumscribed sphere has the radius which goes to the vertex of the cube. Clearly the inscribed radius r is half the edge of the cube, so that one cube vertex has coordinate (r, r, r) if the center of the spheres were the origin. The circumscribed radius is thus $\sqrt{r^2 + r^2 + r^2} = r\sqrt{3}$. The circumscribed volume is then $V = \frac{4}{3}\pi(r\sqrt{3})^3 = \frac{4}{3}\pi(3\sqrt{3})r^3$. The ratio is thus $3\sqrt{3}$.