CSU FRESNO MATHEMATICS FIELD DAY

MAD HATTER MARATHON 11-12
PART I

April 26^{th}, 2014

1. You row upstream at 3 miles per hour and return on the same route at 6 miles per hour. What was your average speed in miles per hour for the whole trip?

   (a) 4 miles per hour
   (b) 4.5 miles per hour
   (c) 3.5 miles per hour
   (d) 4.25 miles per hour
   (e) 5 miles per hour

2. For what value of n is it true that $3^1 \cdot 3^2 \cdot 3^3 \cdots 3^n = 3^{253}$?

(a) $n = 20$
(b) $n = 21$
(c) $n = 22$
(d) $n = 23$
(e) No value of n

3. A group of students takes a test and the average score is 74. If one more student had taken the test and scored 100, the average would have been 74.5. How many students took the test?

   (a) 45
   (b) 47
   (c) 49
   (d) 51
   (e) 53

4. Triangle ABC has $AC = 15, BC = 13$, and $AB = 4$. What is the length of the altitude from C to the extension of AB?

(a) 9
(b) 10
(c) 11
(d) 12
(e) 13

5. Find the last two digits of 2014^{2014}.

(a) 36
(b) 16
(c) 56
(d) 76
(e) 96

6. Each of Alice, Bob, Carol, and Don took a test. Each of them answered at least one question correctly, and altogether they answered 67 questions correctly. Alice had more correct answers than anyone else. Bob and Carol together answered 43 questions correctly. How many correct answers did Don have?

(a) 2
(b) 1
(c) 5
(d) 4
(e) 6

7. During the lunch break from 12 : 00 to 1 : 00, Bill eats, then checks his messages, then goes to the restroom, then talks to a friend. Each activity after the first takes half as much time as the preceding activity. There are no intervening time intervals. At what time did Bill finish checking his messages?

(a) 12 : 36
(b) 12 : 40
(c) 12 : 44
(d) 12 : 48
(e) 12 : 52

8. Suppose S is a set of positive integers, each of which is less than 25, such that no two elements of S have a common divisor greater than 1. What is the largest possible number of elements in S?

(a) 7
(b) 8
(c) 10
(d) 11
(e) 12

9. A room contains 2014 light bulbs with one switch for each bulb. All bulbs are off to begin. A group of 2014 people walk in to the room one at a time and perform the following action:
The first person changes the state of all switches, the second person changes the state of every second switch (switches 2, 4, 6, etc.), the third person changes the state of every third switch (switches 3, 6, 9 etc.), and so on, until the 2014^{th} person changes the state of the 2014^{th} switch.
After the last person has left the room, how many light bulbs are switched on?

(a) 40
(b) 41
(c) 42
(d) 43
(e) 44

10. What is the minimal number of positive divisors of x if $x > 1$, and $x, x^{5/6}$, and $x^{7/8}$ are all integers?

(a) 23
(b) 25
(c) 30
(d) 27
(e) 32

11. What is the number of integers n for which $\frac{7n+15}{n-3}$ is an integer?

(a) 6
(b) 9
(c) 12
(d) 18
(e) 24

12. A quadratic polynomial f satisfies $f(x) \geq 1$ for all $x, f(2) = 1$, and $f(3) = 3$. What is $f(5)$?

(a) 15
(b) 16
(c) 17
(d) 19
(e) 21

13. Tom, Dick, and Harry were playing tennis. After each game, the loser was replaced by the third person. At the end of the day, Tom had played 27 games, and Dick played 13 games. How many games did Harry play?

(a) 11
(b) 12
(c) 13
(d) 14
(e) 15

14. A “hundred-year flood” is defined as a flood of such magnitude that the probability of its occurrence in any given year is 0.01. What is the probability that a 70-year old man has seen at least one hundred-year flood in his lifetime?

(a) $1 - (0.99)^{70}$
(b) $(0.01)^{70}$
(c) $1 - (0.01)^{70}$
(d) 0.70
(e) 1

15. 2014! is divisible by 2^k. What is the largest possible value of k?

(a) 1006
(b) 1509
(c) 1910
(d) 2005
(e) 2010

16. Points A, C, and D lie on a circle. Point B lies outside the circle such that B, D, and C are collinear with D between B and C, and BA is tangent to the circle. If $AB = 2, AC = 3$, and $BD = 1$, what is the area of triangle ABC?

(a) 1
(b) 2
(c) $\frac{3}{4}\sqrt{15}$
(d) $\frac{3}{4}\sqrt{11}$
(e) $2\sqrt{11}$

17. A right circular cone has height equal to radius. What is the ratio of its volume to that of a cube inscribed inside it, with the base of the cube lying on the base of the cone?

(a) $\frac{\pi}{12}(10+\sqrt{2})$
(b) $\frac{\pi}{12}(10+3\sqrt{2})$
(c) $\frac{\pi}{12}(10+5\sqrt{2})$
(d) $\frac{\pi}{12}(10+7\sqrt{2})$
(e) $\frac{\pi}{12}(10+9\sqrt{2})$

18. $ABCD$ is a rectangle in which the length AB minus the length AD equals 10. Inside $ABCD$ is a square $WXYZ$ with sides parallel to those of the rectangle, and W closest to A, and X closest to B. The total of the areas of the trapezoids $XBCY$ and $AWZD$ is 1000, while the total area of the trapezoids $ABXW$ and $ZYCD$ is 400. What is the area of the square $WXYZ$?

(a) 400
(b) 1600
(c) 3600
(d) 4900
(d) 6400

19. How many 10-digit strings of 0's and 1's are there that do not contain any consecutive 0's?

(a) 144
(b) 140
(c) 89
(d) 85
(e) 133

20. Let BE be a median of triangle ABC, and let D be a point on AB such that $BD/DA = 3/7$. What is the ratio of the area of triangle BED to that of triangle ABC?

(a) 3/20
(b) 7/20
(c) 1/5
(d) 1/4
(e) the answer cannot be detemined

21. You write five letters to different people, and address the corresponding envelopes. In how many ways can the letters be placed in the envelopes, with one letter in each envelope, so that none of them is in the correct envelope?

(a) 36
(b) 40
(c) 42
(d) 44
(e) 52

22. An equilateral triangle in the first quadrant has vertices at the points $(0,0),(x_1,4)$, and $(x_2,11)$. What is the ordered pair (x_1,x_2)?

(a) $(\sqrt{3},6\sqrt{3})$
(b) $(\sqrt{3},\sqrt{3})$
(c) $(6\sqrt{3},\sqrt{3})$
(d) $(6\sqrt{3},6\sqrt{3})$
(e) $(3\sqrt{6},\sqrt{6})$

23. The diagonals of a parallelogram partition it into four triangles. Let G be the centroid of one of the triangles, and let T be a triangle formed by G and two vertices of the parallelogram. What is the largest possible ratio of the area of T to that of the parallelogram?

(a) $3/4$
(b) $2/3$
(c) $7/12$
(d) $1/2$
(e) $5/12$

24. How many ordered 4-tuples of non-negative integers (a, b, c, d) satisfy $a + b + c + d \leq 15$?

(a) 3524
(b) 3672
(c) 3716
(d) 3876
(e) none of the above

25. Out of all relatively prime integers a and b, what is the largest possible value of the greatest common divisor of $a + 201b$ and $201a + b$?

(a) 37542
(b) 39264
(c) 40400
(d) 42176
(e) 44862

26. Each of two urns contains N balls. All balls are either red or black, and each urn contains at least one red ball and one black ball. You randomly select and urn and a ball from it, and then put the ball back. Then you do this again. What is the smallest value of N for which it is possible that the probability that you chose two red balls from the first urn equals the probability that you chose two red balls or two black balls from the second urn?

(a) 9
(b) 4
(c) 12
(d) 6
(e) 7

27. Let $f(x) = x^2 + 10x + 20$. For what real values of x is $f(f(f(f(x)))) = 0$?

(a) $\pm 5^{1/4} - 5$
(b) $\pm 5^{1/8} - 5$
(c) $\pm 5^{1/10} - 5$
(d) $\pm 5^{1/12} - 5$
(e) $\pm 5^{1/16} - 5$

28. In the standard version of tic-tac-toe, two players X and O alternately fill in a 3×3 board with their symbols, with X moving first. The game stops when one of the two players has three of his/her symbols in the same row, column, or diagonal, or all squares are filled. How many ways of filling in the entire board with X's and O's can be the end result of a valid game of tic-tac-toe?

(a) 78
(b) 82
(c) 86
(d) 100
(e) 122.

29. Find the sum of all positive integers n with no more than 3 digits for which the number obtained as the last three digits of n^2 equals n.

(a) 973
(b) 986
(c) 1002
(d) 1012
(e) 1013

30. What is the value of $xy + yz + zx$ if $x^2 + xy + y^2 = 2, y^2 + yz + z^2 = 1$, and $z^2 + zx + x^2 = 3$, with x, y, and z all positive?

(a) $\frac{2}{3}\sqrt{2}$
(b) $\frac{2}{3}\sqrt{6}$
(c) $\frac{2}{3}\sqrt{7}$
(d) $\sqrt{3}$
(e) none of the above

Solutions

1. A
2. C
3. D
4. D
5. B
6. B
7. D
8. C
9. E
10. B
11. D
12. D
13. D
14. A
15. D
16. C

17. D
18. C
19. A
20. A
21. D
22. C
23. E
24. D
25. C
26. E
27. E
28. A
29. C
30. B