CSU FRESNO MATHEMATICS FIELD DAY

MAD HATTER MARATHON 11-12
PART II

April 26^{th}, 2014

1. Evaluate $\ln(\tan 1^\circ) + \ln(\tan 2^\circ) + \cdots + \ln(\tan 89^\circ)$.

(a) 0
(b) 1
(c) 2
(d) $\ln \pi$
(e) e

2. What is the sum of the digits of $(1010101)^2$?

(a) 12
(b) 16
(c) 17
(d) 19
(e) 21

3. A student takes three exams. The second has twice as many questions as the first, and the third has three times as many questions as the first. The student answers exactly 75% of the questions correctly on the first exam, exactly 81% on the second exam, and exactly 85% on the third exam. Out of all the questions on the three exams, what percent did he answer correctly?

(a) 78%
(b) 79%
(c) 81%
(d) 82%
(e) 83%

4. A sequence satisfies $a_1 = 3, a_2 = 5$, and $a_{n+2} = a_{n+1} - a_n$ for $n \geq 1$. What is the value of a_{2014}?

(a) 2
(b) 3
(c) 5
(d) -5
(e) -3

5. The set of points (x, y) which satisfy $|x| \leq 1, |y| \leq 1$, and $|y| = |x| + x$ consists of several line segments. What is the sum of the lengths of these segments?

(a) 2
(b) $1 + \sqrt{3}$
(c) $1 + \sqrt{5}$
(d) 3
(e) $2 - \sqrt{3}$

6. Suppose that $f(x)$ is defined for all values $x > 0$ and suppose that $f(x) + f(1/x) = 0$ for all $x > 0$. What is $f(2)$?

(a) 0
(b) 1
(c) 2
(d) 3
(e) cannot be determined

7. On Saturday, four trucks drove in a straight line (one directly behind the other). None of the trucks passed any of the others, so the order of the trucks never changed. How many ways are there to rearrange the order of the trucks so that on Sunday, no truck is directly behind a truck that it was directly behind on Saturday?

(a) 4
(b) 8
(c) 11
(d) 12
(e) 16

8. How many positive integers less than 1200 have no repeating digits; i.e., no digit occurs more than once.

(a) 648
(b) 682
(c) 794
(d) 812
(e) 817

9. What is the value of the sum

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{99}{100!}?$$

(a) $\frac{1}{2}-\frac{1}{100!}$

(b) $1-\frac{1}{100!}$

(c) $1-\frac{1}{99!}$

(d) $\frac{1}{2}-\frac{1}{99!}$

(e) 1

10. The sum of the first ten terms of a nonzero geometric series is 244 times the sum of the first five terms. What is the common ratio?

(a) 2
(b) 5/2
(c) 3
(d) 7/2
(e) 4

11. Five circles of equal radius are placed inside a square of side length 1 in such a way that no two intersect in more than one point. What is the largest possible radius for these circles?

(a) 1
(b) $\frac{1}{2}\sqrt{2}$
(c) $\frac{1}{3}\sqrt{3}$
(d) $\frac{1}{2}(\sqrt{2}-1)$
(e) $\frac{1}{3}(\sqrt{3}-1)$

12. You take six steps, each time moving either one step forward or one step back, with probability $\frac{1}{2}$ of each each time. What is the probability that you end up back where you started?

(a) 1/2
(b) 1/4
(c) 5/16
(d) 2/3
(e) 7/16

13. If $x = 2 + \sqrt{3}$, what is the value of $x^4 + \frac{1}{x^4}$?

(a) 192
(b) 194
(c) 196
(d) 198
(e) 200

14. For $n \geq 1$, let d_n denote the length of the line segment connecting the two points where the line $y = x + n + 1$ intersects the parabola $8x^2 = y - \frac{1}{32}$. What is the value of the sum $\displaystyle\sum_{n=1}^{1000} \frac{1}{n \cdot d_n^2}$?

(a) $\frac{1}{999}$
(b) $\frac{1000}{1001}$
(c) $\frac{998}{999}$
(d) $\frac{1}{1001}$
(e) $\frac{999}{1000}$

15. Let $P(x)$ be a monic polynomial of degree 3. (*Monic* here means that the coefficient of x^3 is 1.) Suppose that the remainder when $P(x)$ is divided by $x^2 - 5x + 6$ equals two times the remainder when $P(x)$ is divided by $x^2 - 5x + 4$. If $P(0) = 100$, what is $P(5)$?

(a) 99
(b) 106
(c) 110
(d) 124
(e) Cannot be determined

16. Let $P(x)$ be a polynomial of degree 10 such that $P(2^i) = i$ for $0 \leq i \leq 10$. What is the coefficient of x in $P(x)$?

(a) $\frac{511}{256}$
(b) $\frac{1023}{512}$
(c) $\frac{255}{128}$
(d) $\frac{127}{64}$
(e) 1

17. What is the coefficient of x^{50} in the expansion of the following product?

$$(1 + 2x + 3x^2 + 4x^3 + \cdots + 101x^{100}) \cdot (1 + x + x^2 + x^3 + \cdots + x^{25})$$

(a) 50
(b) 125
(c) 501
(d) 923
(e) 1001

18. What is the largest possible value of the function $5\sin x + 12\cos x$?

(a) 12
(b) 13
(c) 17
(d) 14
(e) $\sqrt{119}$

19. Let $P_n = 1^n + 2^n + 3^n + 4^n$. Find the number of integers n for which $1 \leq n \leq 100$ and P_n is a multiple of 5.

(a) 68
(b) 75
(c) 86
(d) 98
(e) 100

20. If n is a positive integer, let $r(n)$ denote the number obtained by reversing the order of the digits of n. For example, $r(16) = 61$. For how many two-digit positive integers n is $n + r(n)$ a square of a positive integer?

(a) 2
(b) 4
(c) 6
(d) 8
(e) 10

21. A Fibonacci-like sequence of numbers is defined by $a_1 = 1, a_2 = 3$, and for $n \geq 3, a_n = a_{n-1} + a_{n-2}$. One can compute that $a_{29} = 1149851$ and $a_{30} = 1860498$. What is the value of $\sum_{n=1}^{28} a_n$?

(a) 1149851
(b) 1860498
(c) 1149848
(d) 1860495
(e) None of the above

22. Let $p(x)$ be a polynomial of degree 4 satisfying $p(2) = p(-2) = p(-3) = 1$ and $p(1) = p(-1) = 2$. What is $p(0)$?

(a) 1
(b) 5/2
(c) 3
(d) 9/2
(e) 5

23. What is the length of the shortest path that begins at the point $(2, 5)$, touches the x-axis and then ends at a point on the circle $(x + 6)^2 + (y - 10)^2 = 16$?

(a) 12
(b) 13
(c) $4\sqrt{10}$
(d) $6\sqrt{5}$
(e) $4 + \sqrt{89}$

24. In how many ways can the faces of a cube be colored using six colors, if each face is to be a different color, and two colorings are considered the same when one can be obtained from the other by rotating the cube?

(a) 5
(b) 10
(c) 20
(d) 30
(e) 40

25. A *Fibo* sequence $a_1, a_2, \ldots$ is one in which a_1 and a_2 are positive integers, and $a_n = a_{n-2} + a_{n-1}$ for $n \geq 3$. If $a_1, a_2, \ldots, 81, \ldots$ is the Fibo sequence which contains 81 and has the largest possible number of terms preceding the 81, what is the value of $a_1 + a_2$?

(a) 2
(b) 5
(c) 3
(d) 7
(e) 10

26. Which of the following is largest (all angles are in degrees)?

(a) $\sin(45^\circ) + \cos(45^\circ)$
(b) $\sin(60^\circ) + \cos(60^\circ)$
(c) $\sin(90^\circ) + \cos(90^\circ)$
(d) $\sin(120^\circ) + \cos(120^\circ)$
(e) $\sin(135^\circ) + \cos(135^\circ)$

27. Given that $29a031 \times 342 = 100900b02$ where a, b denote missing digits, what is the value of $a + b$?

(a) 7
(b) 8
(c) 9
(d) 10
(e) 11

28. A person rowing a boat against a river current got from point A to point B in 4 hours. It took him 2 hours to get from point B to point A rowing along the current. How long would it take him to row the same distance as between A and B on a lake with no current?

(a) 2h 30 min
(b) 2h 40 min
(c) 3h
(d) 3h 15 min
(e) 3h 20 min

29. A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the x-intercepts of these lines?

(a) 4
(b) 6
(c) 8
(d) 10
(e) 12

30. Compute $S = \frac{1}{5} + \frac{1}{25} + \frac{2}{125} + \frac{3}{625}\frac{5}{3125} + \cdots$, where each numerator is the sum of the two preceding numerators, and each denominator is 5 times the preceding one.

(a) 2/3
(b) 5/19
(c) 7/31
(d) 4/9
(e) 1

Solutions

1. A
2. B
3. D
4. E
5. C
6. E
7. C
8. C
9. B
10. C
11. D
12. C
13. B
14. B
15. C
16. B

17. E
18. B
19. B
20. D
21. D
22. B
23. B
24. D
25. B
26. A
27. E
28. B
29. D
30. B