# 2014 <br> Leap Frog Relay Grades 9-10 <br> Part I Solutions 

## No calculators allowed <br> Correct Answer $=4$, Incorrect Answer $=-1$, Blank $=0$

1. The sum of the prime divisors of 2014 is . . .
(a) 76
(b) 78
(c) 80
(d) 82
(e) None of these

Solution. (e) Factor, $2014=2 \times 19 \times 53$. Since 2, 19 and 53 are primes, the sum of the prime divisors of 2014 is

$$
2+19+53=74
$$

2. In $\triangle A B C$, the base length $A B$ is 10 feet and the length $A D$ is $x$ feet. What is the value of $x$ so that the area enclosed by $\triangle A D C$ is twice the area enclosed by $\triangle B D C$ ?

(a) $x=8$.
(b) $x=\frac{9}{2}$
(c) $x=5$
(d) $x=\frac{20}{3}$
(e) None of these

Solution. (d) Draw the altitude of $\triangle A D C$ whose length we'll call $h$. Note that $h$ is also the altitude of $\triangle B D C$. Also note that the length of $B D$ is $10-x$. So, we have

Area $\triangle A D C=\frac{1}{2} x h \quad$ and $\quad$ Area $\triangle B D C=\frac{1}{2}(10-x) h$.


Then, the requirement that $\triangle A D C$ has twice the area of $\triangle B D C$ means

$$
\frac{1}{2} x h=2 \cdot \frac{1}{2}(10-x) h .
$$

Solve this equation for $x$ to get $x=\frac{20}{3}$.
3. The Antarctic Shoe Store is running a $30 \%$ off sale. Sales tax is $10 \%$. The pre-sale price of a pair of shoes is $\$ 78$. When you pay for your pair of shoes, you hand the cashier a $\$ 100$ bill. How much change do you get back?
(a) $\$ 39.82$
(b) $\$ 39.86$
(c) $\$ 39.90$
(d) $\$ 39.94$
(e) None of these

Solution. (d) Thirty percent off of 78 is $0.7 \times 78=54.60$. Add in the $10 \%$ sales tax, $54.60+5.46=60.06$. The change you get back is $\$ 100-\$ 60.06=\$ 39.94$.
4. A cube of ice has melting so that its surface area has decreased by $19 \%$. Assuming at all times, the cube maintains length $=$ width $=$ height, by what percentage has the volume decreased?
(a) $26.7 \%$
(b) $26.9 \%$
(c) $27.1 \%$
(d) $27.3 \%$
(e) None of these

Solution. (c) Denote the side length of the cube by $x$ before melting and by $y$ after melting. Then the two surfaces areas are $6 x^{2}$ and $6 y^{2}$, respectively. We are assuming $6 y^{2}=(1-.19) 6 x^{2}=.81 \times 6 x^{2}$. This implies $y=\sqrt{.81} x=.9 x$. Cubing each side gives us the volume, $y^{3}=.729 x^{3}$. So the volume has decreased by a factor of $1-.729=.271$. This means, the volume has decreased by $27.1 \%$.
5. The 9 -digit number $N=1234 d 4321$ is divisible by 9 . What is the value of the digit $d$ ?
(a) 7
(b) 6
(c) 5
(d) 4
(e) None of these

Solution. (a) A number is divisible by 9 precisely when its digits add up to a multiple of 9 . The sum of the digits of $N$ is

$$
1+2+3+4+d+4+3+2+1=20+d
$$

In order for $20+d$ to be divisible by 9 , we must have $d=7$. And indeed, $123474321=9 \times 13719369$
6. In the figure below $A B C D$ is a rectangle with indicated side lengths. Also, $\overline{B F} \perp \overline{A E}$. Find the length of $B F$.

(a) $B F=2 \sqrt{3}$
(b) $B F=\sqrt{5}$
(c) $B F=\frac{5}{2}$
(d) $B F=\frac{3 \sqrt{2}}{2}$
(e) None of these

Solution. (d) Since $\mathrm{m} \angle A B F=90-\mathrm{m} \angle B A F=\mathrm{m} \angle D A E$, we have similar triangles $\triangle B A F \sim \triangle A E D$. Hence,

$$
\frac{B F}{A B}=\frac{A D}{E A}
$$

Now, $\mathrm{AB}=3, \mathrm{AD}=2$ and by the Pythagorean Theorem applied to $\triangle A D E$ we have $E A=2 \sqrt{2}$. This gives us,

$$
\frac{B F}{A B}=\frac{A D}{E A} \Longrightarrow \frac{B F}{3}=\frac{2}{2 \sqrt{2}} \Longrightarrow B F=\frac{3 \sqrt{2}}{2} .
$$

7. Find the value of $a$ so that the two lines $2013 x+2012 y=1$ and $2014 x+a y=1$ are mutually perpendicular.
(a) $-\frac{2027091}{1009}$
(b) $-\frac{2027091}{1008}$
(c) $-\frac{2027091}{1007}$
(d) $-\frac{2027091}{1006}$
(e) None of these

Solution. (d) The slopes of mutually perpendicular lines are negative reciprocals of one another. We find the slopes by solving for $y$ in each
line.

$$
\begin{aligned}
& \text { Line }_{1}: 2013 x+2012 y=1 \Longrightarrow y=-\frac{2013}{2012} x+\frac{1}{2012} \\
& \text { Line }_{2}: \quad 2014 x+a y=1 \Longrightarrow y=-\frac{2014}{a} x+\frac{1}{a}
\end{aligned}
$$

The respective slopes are then $m_{1}=-2013 / 2012$ and $m_{2}=-2014 / a$. Since the lines are mutually perpendicular, we must have $m_{1}=-1 / m_{2}$,

$$
-\frac{2013}{2012}=\frac{a}{2014} .
$$

Solving for $a$, gives us

$$
\begin{aligned}
a & =-\frac{2013 \times 2014}{2012} \\
& =-\frac{2013 \times 1007}{1006} \\
& =-\frac{2027091}{1006} .
\end{aligned}
$$

8. A standard calendar year has 365 days. A leap year has 366 days. A year is a leap year if it is divisible by 4 , except if it is a new century year not divisible by 400 . So 1900 was not a leap year (1900 is not divisible by 400 ), but 2000 was a leap year ( 2000 is divisible by 400). December 25, 2013 was a Wednesday. What day of the week will December 25 be in the year 3013 ?
(a) Friday
(b) Saturday
(c) Sunday
(d) Monday
(e) None of these

Solution. (b) If we divide 365 by 7 , we get a remainder of 1 . This means that without leap years, the days of the week for a particular
calendar date would advance 1 day per year. There is 1000 years from 2013 to 3013 , so this would account for 1000 added days. However, we must then add the 250 leap days (1000/4) less the non-leap days in the new century years not divisible by 400 (there are 8 of these), giving a total of $1000+250-8=1242$. Divide 1242 by 7 to get a remainder of 3. This means December 25, 3013 will occur 3 days after Wednesday, which is Saturday.
9. Lenny asks Rodney to choose a number. Lenny then doubles Rodney's number and subtracts 6 . Lenny repeats this procedure 2 more times (for a total of 3 times) using the result of the previous calculation as the starting number with each repeat. Lenny then notes that the end result is the number 2014. What was Rodney's starting number?
(a) 253
(b) 254
(c) 255
(d) 256
(e) None of these

Solution. (e) Let's let $x$ be Rodney's original number. Then the result of the 3 repeated procedures is

$$
2(2(2 x-6)-6)-6=8 x-42
$$

Solve the equation $8 x-42=2014$ to get $x=257$, none of the answer choices provided.
10. Lenny has 45 coins (nickels, dimes and quarters) in his purse that add up to $\$ 4$. The Magic Fairy instantly switches the respective numbers of nickels and quarters, doubling the amount of money in Lenny's purse, giving him $\$ 8$. How many quarters did Lenny originally have?
(a) 8
(b) 7
(c) 6
(d) 5
(e) None of these

Solution. (d) Let $n, d$ and $q$ be the original number of nickels, dimes and quarters in Lenny's purse. Since there are 45 coins, we have our first equation:

$$
\mathrm{Eq}_{1}: n+d+q=45
$$

The dollar amount is $\$ 4$, so this gives us another equation $5 n+10 d+$ $25 q=400$. Divide by 5 to get equation number 2 ,

$$
\mathrm{Eq}_{2}: n+2 d+5 q=80
$$

After, the switch, we have $25 n+10 d+5 q=800$ and upon dividing by 5 , we have

$$
\mathrm{Eq}_{3}: 5 n+2 d+q=160 .
$$

Solve the above three equations in 3 unknowns, to get $n=25, d=15$ and $q=5$. So, Lenny originally had 5 quarters.

