2013
Leap Frog Relay Grades 9-10
Part II Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = -1, Blank = 0

1. A music player has a list price of \$100. However, the store is having a 10% off sale for the month of April. But you are in luck, because you came on a Tuesday in April when the store gives an additional 15% *off the sale price* at the register. Assuming sales tax is 10% of the register price, how much are you going to pay for the music player?

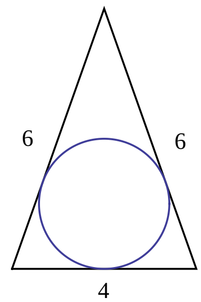
   (a) \$83.85 (b) \$83.95

   (c) \$84.05 (d) \$84.15

   (e) None of these

   Solution. (d) The April sale price for the player is \$90. The register price will then be 85% of \$90, which is \$76.50. Sales tax (10%) on \$76.50 is \$7.65. So you will pay the sum of \$76.50 and \$7.65, which is \$84.15.

2. The circle is inscribed in the isosceles triangle with respective side lengths 6, 6 and 4. Determine the area of the inscribed circle.



(a) $\frac{\pi}{2}$

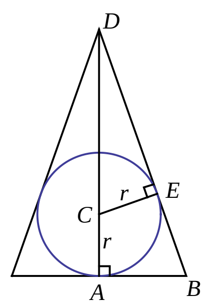
(b) $\frac{3\pi}{2}$

(c) $\frac{5\pi}{2}$

(d) $\frac{7\pi}{2}$

(e) None of these

Solution. (e) Label the figure as indicated below.



We have $2 = AB = EB$, so $DE = 4$. Also, by the Pythagorean Theorem applied to the right triangle $\triangle DAB$, we have $DA = 4\sqrt{2}$. The pair of similar triangles $\triangle ECD \sim \triangle ABD$ implies proportional sides $DA/DE = AB/CE \implies 4\sqrt{2}/4 = 2/r \implies r = \sqrt{2}$. Thus, the area of the circle is $\pi r^2 = 2\pi$, none of the answer choices provided.

3. How many 4-digit palindromic numbers $abba$ are divisible by 9?

(a) 7

(b) 8

(c) 9

(d) 10

(e) None of these

Solution. **(d)** A number is divisible by 9 precisely when its digit sum is divisible by 9. So $abba$ is divisible by 9 when $2a + 2b$ is divisible by 9. Thus, we must have $a + b$ is divisible by 9. Since a and b are digits, we must have $a + b = 9$ or $a + b = 18$. And, since $a \neq 0$, we have the 9 solutions $(a, b) = (1, 8), (2, 7), \ldots (8, 1), (9, 0)$ along with the additional solution $(a, b) = (9, 9)$. This gives us a total of 10 solutions.

4. The two lines $y = 2x + b$ and $y = x + 2013$ meet at a point on the line $y = 4x + 21$. Determine the value of b.

(a) $b = 1344$ (b) $b = 1349$

(c) $b = 1354$ (d) $b = 1359$

(e) None of these

Solution. **(b)** First we find the point of intersection by solving the system $y = x + 2013$ and $y = 4x + 21$.

$$x + 2013 = 4x + 21 \Longrightarrow x = 664 \Longrightarrow y = 2677,$$

giving the point $(664, 2677)$. We substitute this into the equation $y = 2x + b$ and solve for b.

$$2677 = 2(664) + b \Longrightarrow b = 1349.$$

5. What is the volume of the cube that is inscribed in a sphere whose radius is 6 feet?

(a) $188\sqrt{3}$ ft^3 (b) $190\sqrt{3}$ ft^3

(c) $192\sqrt{3}$ ft^3 (d) $194\sqrt{3}$ ft^3

(e) None of these

Solution. **(c)** The diagonal (longest) length in the cube is twice the sphere radius, $2 \times 6 = 12$ feet. If we call the (equal) side lengths of the cube x feet, then we have, by the Pythagorean Theorem, $3x^2 = 12^2 = 144$, so $x = \sqrt{48} = 4\sqrt{3}$ feet. The volume of the box is then $x^3 = (4\sqrt{3})^3 = 192\sqrt{3}$ cubic feet.

6. Ten *consecutive* natural numbers sum to 1005. What is the sum of the smallest and largest of these ten natural numbers? (A consecutive list of numbers is in the from $n, n+1, n+2, \ldots$.)

   (a) 201
   (b) 203
   (c) 205
   (d) 207
   (e) None of these

   Solution. (a) Let the smallest natural number be n. Then the sum of the ten is

$$1005 = n + (n+1) + \cdots + (n+9) = 10n + 45.$$

   Solve for n to get $n = 96$. So the sum of the smallest and largest is

$$n + n + 9 = 96 + 96 + 9 = 201.$$

7. If $4^{x+1} = 8^{2x+3}$, then $16^x = \ldots$.

   (a) $\frac{1}{2}$
   (b) 256
   (c) $\sqrt{2}$
   (d) $\frac{1}{128}$
   (e) None of these

   Solution. (d) Convert the two exponentials to base 2, $4^{x+1} = (2^2)^{x+1} = 2^{2x+2}$ and $8^{2x+3} = (2^3)^{2x+3} = 2^{6x+9}$. Then,

$$\begin{aligned} 4^{x+1} = 8^{2x+3} &\implies 2^{2x+2} = 2^{6x+9} \\ &\implies 2x + 2 = 6x + 9 \\ &\implies x = -7/4. \end{aligned}$$

   Now compute 16^x.

$$\begin{aligned} 16^x &= 16^{-7/4} \\ &= \frac{1}{16^{7/4}} \\ &= \frac{1}{2^7} \\ &= \frac{1}{128}. \end{aligned}$$

8. Find the real number solution to the equation

$$\frac{1}{x} + \frac{x}{3} = \frac{1+x}{x+3}.$$

(a) $x = 1 - \sqrt[3]{9}$

(b) $x = -\sqrt[3]{6}$

(c) $x = 1 - \sqrt[3]{6}$

(d) $x = -\sqrt[3]{9}$

(e) None of these

Solution. (d)

$$\begin{aligned}
\frac{1}{x} + \frac{x}{3} = \frac{1+x}{x+3} \quad &\Longrightarrow \quad \frac{3+x^2}{3x} = \frac{1+x}{x+3} \\
&\Longrightarrow \quad (3+x^2)(x+3) = 3x(1+x) \\
&\Longrightarrow \quad x^3 + 3x^2 + 3x + 9 = 3x^2 + 3x \\
&\Longrightarrow \quad x^3 + 9 = 0 \\
&\Longrightarrow \quad x = -\sqrt[3]{9}.
\end{aligned}$$

9. The graph of the parabola $y = ax^2 + bx + c$ goes through the point $(-1, 3)$ and has vertex $(1, 1)$. Compute the product abc.

(a) $abc = -\frac{3}{4}$

(b) $abc = -\frac{5}{4}$

(c) $abc = -\frac{7}{4}$

(d) $abc = -\frac{9}{4}$

(e) None of these

Solution. (a) Since the graph goes through the point $(1, 1)$, we have that

$$1 = a + b + c.$$

And, the x-coordinate of the vertex is $-b/2a$, which tells us

$$1 = -b/2a \Longrightarrow b = -2a.$$

Substitute $b = -2a$ into $1 = a + b + c$ to get

$$1 = -a + c.$$

Finally, since the graph goes through the point $(-1, 3)$, we have $3 = a - b + c$. If we substitute $b = -2a$ into this equation, we get

$$3 = 3a + c.$$

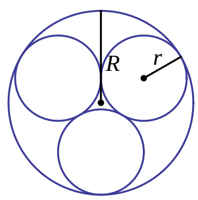
Solve the above two equations in a and c, obtaining

$$a = 1/2 \quad \text{and} \quad c = 3/2.$$

We can now determine $b = -2a = -2(1/2) = -1$. Thus,

$$abc = \frac{1}{2} \cdot (-1) \cdot \frac{3}{2} = -\frac{3}{4}.$$

10. In the figure below, the three small circles all have the same radius r and are mutually tangent to each other, as well as tangent to the larger circle with radius R. Then, $R/r = \ldots$.

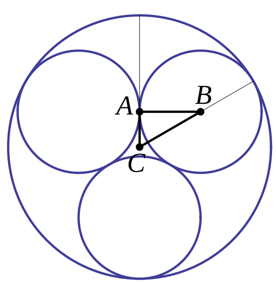


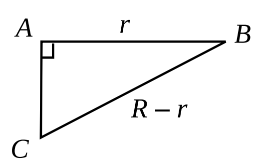
(a) $(3 + \sqrt{3})/2$.　　(b) $(3 + 2\sqrt{3})/2$.

(c) $(2 + 3\sqrt{3})/3$.　　(d) $(3 + 2\sqrt{3})/3$.

(e) None of these

Solution. **(d)** We'll focus on the $30^\circ - 60^\circ - 90^\circ$ triangle $\triangle ABC$.





Because $m\angle ABC = 30^\circ$, we know $BC/AB = 2/\sqrt{3}$,

$$\frac{R-r}{r} = \frac{2}{\sqrt{3}} \implies R\sqrt{3} - r\sqrt{3} = 2r$$
$$\implies \frac{R}{r} = \frac{2+\sqrt{3}}{\sqrt{3}}$$
$$\implies \frac{R}{r} = \frac{3+2\sqrt{3}}{3}.$$