

## MATH 149

# Study Guide and Sample Problems for Test 1

Note: the actual test will consist of five questions, some of which will be computational, some will ask for a brief explanation, and some will require a rigorous detailed proof. Some of the problems will be very similar to homework problems and/or those discussed in class, but some will be different. So make sure that you understand well all the concepts discussed, know precise definitions and basic properties, rather than memorize how to solve specific problems.

### 1. Divisibility and congruences

- State and prove divisibility tests for 2, 3, 4, 5, 8, 9, 10.
- Prove that if  $a|b$  and  $a|c$ , then  $a|(b + c)$ .
- Prove that  $a \equiv b \pmod{10}$  if and only if  $a$  and  $b$  have the same units digit.

### 2. Bases

- When a positive integer  $A$  is written in base 5, it has exactly the same digits as a positive integer  $B$  written in base 6. Which is larger,  $A$  or  $B$ ?
- An integer  $x$  has 3 digits when written in base 10. How many digits can it have when written in base 3?
- Convert  $2000_4$  to base 7.

### 3. Combinatorics

- Explain why there are  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  ways to choose  $k$  objects out of  $n$ .
- Prove that  $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$ .

### 4. Binomial theorem

- Expand:  $(x + y)^n$ .
- Find the first three terms in the expansions of  $(2x + 1)^5$ ,  $(x - 2y)^{10}$ .

### 5. Probability

- There are 5 white, 6 red, and 7 blue balls in a bag. Two balls are drawn randomly. What is the probability that they are both blue?
- There are 5 white, 6 red, and 7 blue balls in a bag. Two balls are drawn randomly. Neither of them is white. What is the probability that they are both blue?
- How many ways are there to choose 3 cards from a deck of 52 cards? How many ways are there to choose 3 cards from the 12 face cards (J, Q, K)?
- If three cards are chosen randomly for a deck of 52 cards, what is the proba-

bility that all three are face cards?

6. Proportional reasoning, ratios, and rates

- (a) If  $5x = 6y$ , what is the ratio of  $x$  to  $y$ ?
- (b) The distance fallen by a freely falling body is proportional to the square of the time it has been falling. If an object falls 144 feet in 3 seconds, how far will it fall in 6 seconds?
- (c) If Misha can eat a large box of oatmeal in two months and Katya can eat the same box in one month, in how much time will they eat one box together?
- (d) Luke can solve 5 problems in 10 minutes. How many problems can he solve in 30 minutes? Five hours? One week?

7. Geometry

- (a) Give the definition and some important properties of similar figures/solids.
- (b) Two rectangles are similar with a scale factor of 3. One of them has area 24 and the other has perimeter 18. Find the dimensions of both rectangles.
- (c) What is the radius of a sphere whose volume is twice larger than that of a sphere of radius  $r$  ?
- (d) The area of a circle of radius  $r$  is  $\pi r^2$  and the circumference is  $2\pi r$ . Observe that  $(\pi r^2)' = 2\pi r$ . Explain why this is not a coincidence.
- (e) It can be shown (e.g. using integration) that the 4-dimensional hypervolume of a sphere in  $R^4$  (i.e. the hypervolume of the solid given by  $x^2 + y^2 + z^2 + w^2 \leq r^2$ ) is  $\frac{1}{2}\pi^2 r^4$ . What is its 3-dimensional “surface” hyperarea?