2013
Leap Frog Relay Grades 9-10
Part I Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = -1, Blank = 0

1. If 5 forks balance 3 knives and 4 knives balance 7 spoons, how many forks balance 21 spoons?

   (a) 19 forks
   (b) 20 forks
   (c) 21 forks
   (d) 22 forks
   (e) None of these

   Solution. (b) We can write the balances as equations. Choose a unit of weight, say ounces, and let F, K and S denote the respective weights of a fork, knife and spoon in ounces. Then we are given $5F = 3K$ and $4K = 7S$. If we multiply the second equation by 3, we obtain $12K = 21S$. So we multiply the first equation by 4 to get $20F = 12K$. Combining these last two equations, we see $20F = 12K = 21S$. So 20 forks will balance 21 spoons.

2. Five students take a 100 point test. Their average score is 82. Suppose a sixth student takes the test and as a result increases the average (of all six scores) by 2 points. What was the test score of the sixth student?

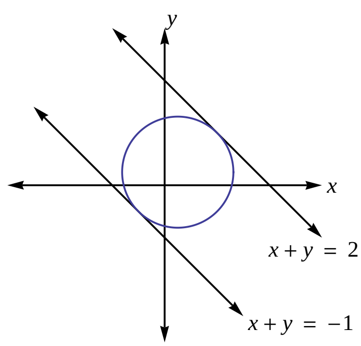
(a) 92 (b) 94

(c) 96 (d) 98

(e) None of these

Solution. **(b)** Let the respective test scores of the six students be $S_1, \ldots, S_6$. We are given that $(S_1 + \cdots + S_5)/5 = 82$, which means $S_1 + \cdots + S_5 = 410$. We are also given that $(S_1 + \cdots + S_5 + S_6)/6 = 82 + 2 = 84$, which means $S_1 + \cdots + S_5 + S_6 = 504$. Now, substitute $S_1 + \cdots + S_5 = 410$ into the last equation to get $410 + S_6 = 504$. Then solve for $S_6 = 94$.

3. The circle in the figure is mutually tangent to the two lines $x + y = 2$ and $x + y = -1$. What is the area enclosed by the circle?



(a) $\frac{5\pi}{8}$ (b) $\frac{7\pi}{8}$

(c) $\frac{9\pi}{8}$ (d) $\frac{11\pi}{8}$

(e) None of these

Solution. **(c)** Using symmetry, we can see that the circle touches the two lines at the two points $A = (1, 1)$ and $B = (-1/2, -1/2)$, respectively. And since the two lines are parallel, the radius, r, of the circle is half the distance between A and B.

$$r = \frac{1}{2}\sqrt{\left(1 + \frac{1}{2}\right)^2 + \left(1 + \frac{1}{2}\right)^2}$$

$$= \frac{3}{4}\sqrt{2}.$$

So, the area enclosed by the circle is

$$\begin{aligned} \pi r^2 &= \pi\left(\frac{3}{4}\sqrt{2}\right)^2 \\ &= \frac{9\pi}{8}. \end{aligned}$$

4. Move forward 3 steps, backward 6 steps, forward 9 steps, etc. If you continue this way forward/backward in multiples of 3, ending with a final forward 2013 steps, how far are you (in steps) from your original position?

(a) 999 steps.

(b) 1002 steps.

(c) 1005 steps.

(d) 1008 steps.

(e) None of these

Solution. (d) Put the dance on the number line, with the real number x representing your place on the line. Let the origin ($x_0 = 0$) be the starting point. The first forward/backward (F 3/B 6) places you at $x_1 = -3$. The second (F 9/B 12) places you at $x_2 = -6$ and the third (F 15/B 18) places you at $x_3 = -9$. We can see the pattern, the nth (F $6n - 3$/B $6n$) places you at $x_n = -3n$. So, the 335th (F 2007/B 2010) places you at $x_{335} = -3 \cdot 335 = -1005$. Your final forward movement of 2013 steps places you at $x_{335} + 2013 = -1005 + 2013 = 1008$.

5. How many real number solutions to the equation

$$x^{2013} - 3x^{2011} = x^{2011} - 3x^{2009}$$

are there?

(a) 2

(b) 3

(c) 4

(d) 5

(e) None of these

Solution. **(d)** Factor both sides of the equation:

$$x^{2011}(x^2 - 3) = x^{2009}(x^2 - 3).$$

If $x = 0$, we clearly get a solution. Now suppose $x \neq 0$, and divide each side of the above equation by x^{2009}, getting

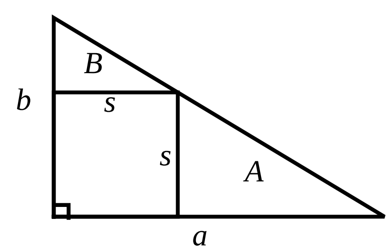
$$x^2(x^2 - 3) = x^2 - 3.$$

If $x = \pm\sqrt{3}$, we obtain two more solutions. Now suppose $x \neq \pm\sqrt{3}$ and divide each side of the above equation by $x^2 - 3$,

$$x^2 = 1.$$

We clearly get two more solutions, $x = \pm 1$. We have a total of 5 solutions: $0, \pm 1, \pm\sqrt{3}$.

6. In the figure below, the large right triangle has respective leg lengths a and b as pictured. The s by s square is inscribed in the triangle. The respective areas of the two smaller right triangles are A and B as indicated. Determine the ratio of the areas A/B as a function of a and b.



(a) $A/B = a^2/b^2$

(b) $A/B = a/b$

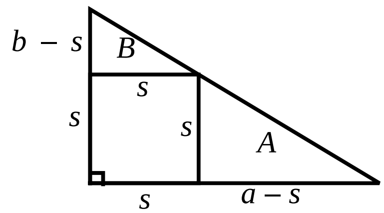
(c) $A/B = (ab)/(a+b)$

(d) $A/B = \sqrt{a^2 + b^2}/(a+b)$

(e) None of these

Solution. **(a)** If we compare the similar triangle pair that is the small top triangle with the large triangle, we get equal ratios

$$\frac{b-s}{s} = \frac{b}{a} \Longrightarrow s = \frac{ab}{a+b}.$$



So,

$$\begin{aligned}\frac{A}{B} &= \frac{\frac{1}{2}(a-s)s}{\frac{1}{2}(b-s)s} \\ &= \frac{a-s}{b-s} \\ &= \frac{a-\frac{ab}{a+b}}{b-\frac{ab}{a+b}} \\ &= \frac{a^2}{b^2}.\end{aligned}$$

7. If the parabola whose equation is $y = ax^2 + bx + c$ goes through the point $(0, 2)$ and has its vertex at $(-1, 1)$, then $a + b + c =$ ________.

(a) 2

(b) 3

(c) 4

(d) 5

(e) None of these

Solution. (d) First substitute the point $(x, y) = (0, 2)$ into the equation $y = ax^2 + bx + c$ to get $c = 2$. Next, use the formula $x = -b/2a$ for the x-coordinate of the vertex to get $-1 = -b/2a \Longrightarrow b = 2a$. Finally, the y-coordinate of the vertex is

$$\begin{aligned}1 &= a(-1)^2 + b(-1) + 2 \\ &= a - 2a + 2 \\ &= -a + 2.\end{aligned}$$

This implies $a = 1$, and so $b = 2$. So, the equation of the parabola is $y = x^2 + 2x + 2$, so $a + b + c = 1 + 2 + 2 = 5$.

8. Suppose a, b, c are real numbers such that $a+b+c=3$ and $ab+ac+bc=2$. Then $a^2+b^2+c^2=$ ________.

(a) 3

(b) 4

(c) 5

(d) 6

(e) None of these

Solution. (c) Square the equation $a+b+c=3$ to get

$$(a+b+c)^2 = 9 \Longrightarrow a^2+b^2+c^2+2(ab+ac+bc) = 9.$$

Now, substitute $ab+ac+bc=2$ in the above equation to get

$$a^2+b^2+c^2+4 = 9 \Longrightarrow a^2+b^2+c^2 = 5.$$

9. If x is 20% of y, then y is ________% of x.

(a) 5

(b) 120

(c) 80

(d) 50

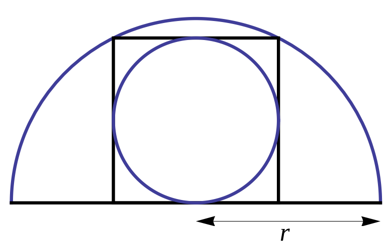
(e) None of these

Solution. (e) If x is 20% of y, then $x=(.2)y$. Solve for y in terms of x,

$$y = \frac{1}{.2}x = \frac{10}{2}x = 5x.$$

So, y is five time x, that is, y is 500% of x, none of the answer choices provided.

10. The semicircle pictured has a radius equal to r inches. The square is inscribed in the semicircle and the smaller circle is inscribed in the square. What is the area of the smaller circle in terms of r?



(a) $\dfrac{\pi r^2}{6}$ inches^2.

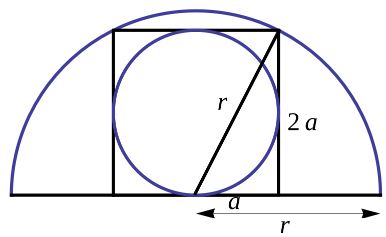
(b) $\dfrac{\pi r^2}{5}$ inches^2.

(c) $\dfrac{\pi r^2}{4}$ inches^2.

(d) $\dfrac{\pi r^2}{3}$ inches^2.

(e) None of these

Solution. **(b)** Let a be the radius of the smaller circle. We can see from the figure below that a, $2a$ and r are the two legs and respective hypotenuse of a right triangle.



So, by the Pythagorean Theorem, we have

$$a^2 + (2a)^2 = r^2 \Longrightarrow 5a^2 = r^2 \Longrightarrow a^2 = \frac{r^2}{5}.$$

Thus, the area of the smaller circle is

$$\frac{\pi r^2}{5} \text{ inches}^2.$$