2014
Leap Frog Relay Grades 9-10
Part II Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

11. The sum of the first 2014 positive odd integers is subtracted from the sum of the first 2014 positive even integers. What is the result?

(a) 1 (b) 0

(c) 4028 (d) 2013

(e) None of these

Solution. **(e)** The sum of the first 2014 positive odd numbers is

$$1 + 3 + 5 + \cdots + (2 \times 2014 - 1).$$

The sum of the first 2014 positive even numbers is

$$2 + 4 + 6 \cdots + (2 \times 2014).$$

The difference is

$$\begin{aligned}(2-1)+(4-3)+(6-5)+\cdots((2\times 2014)-(2\times 2014-1)) &= \underbrace{1+1+\cdots 1}_{2014} \\ &= 2014.\end{aligned}$$

12. The Magic Fairy changed a 5 dollar bill in Lenny's pocket to a 20 dollar bill, tripling the amount of money he had. How much money does Lenny have now?

(a) \$22.50 (b) \$23.50

(c) \$24.50 (d) \$25.50

(e) None of these

Solution. (a) Let's say the amount of money Lenny originally had in his pocket was $5 + x$ dollars. After the Magic Fairy changed the 5 dollar bill to a 20 dollar bill, he now has $20 + x$ dollars. If this is triple the original amount, we have an equation

$$20 + x = 3(5 + x).$$

We solve for $x = 2.5$. So the amount Lenny has now is $20 + 2.5 = 22.5$.

13. Solve the equation for x.

$$\frac{1}{2x} + \frac{3}{4x} + \frac{5}{6x} = 7.$$

(a) $x = \frac{31}{84}$ (b) $x = \frac{29}{84}$

(c) $x = \frac{27}{84}$ (d) $x = \frac{25}{84}$

(e) None of these

Solution. (d) Combine the three fractions using the common denominator $12x$ to get

$$\frac{6 + 9 + 10}{12x} = 7 \Longrightarrow \frac{15}{12x} = 7 \Longrightarrow x = \frac{25}{84}.$$

14. The pair of lines $2x + by = 7$ and $ax + 5y = 6$ are parallel. What is the value of the product ab?

(a) $ab = 10$

(b) $ab = 9$

(c) $ab = 8$

(d) $ab = 7$

(e) None of these

Solution. **(a)** Solve for y from each of the equations.

$$2x + by = 7 \implies y = -\frac{2}{b}x + \frac{7}{b}$$
$$ax + 5y = 6 \implies y = -\frac{a}{5}x + \frac{6}{5}.$$

The lines are parallel precisely when their slopes are equal,

$$-\frac{2}{b} = -\frac{a}{5} \iff ab = 10.$$

15. A rectangle has three vertices $A(-10, 5), B(0, 0), C(6, 12)$. What are the coordinates of the fourth vertex $D(x, y)$ of the rectangle $ABCD$?

(a) $(x, y) = (-3, 16)$

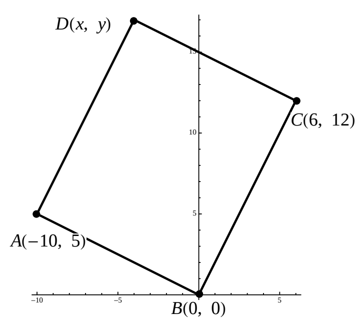
(b) $(x, y) = (-4, 16)$

(c) $(x, y) = (-3, 17)$

(d) $(x, y) = (-4, 17)$

(e) None of these

Solution. **(d)** The slope of line $\overleftrightarrow{AB}$ is $-1/2$. So the equation of line $\overleftrightarrow{CD}$ is $y - 12 = (-1/2)(x - 6)$, which simplifies to $y = (-1/2)x + 15$. Similarly, the equation of line $\overleftrightarrow{AD}$ is $y - 5 = 2(x + 10)$, which simplifies to $y = 2x + 25$.



Now solve the two equations:

$$\begin{aligned} y &= (-1/2)x + 15 \\ y &= 2x + 25, \end{aligned}$$

to get the coordinates of D, which is

$$(x, y) = (-4, 17).$$

16. Lenny has a coin jar of pennies and nickels in a ratio Pennies:Nickles = 3:2. Lenny's friend Marty gives Lenny the 50 pennies he (Marty) has in his pocket to add to Lenny's coin jar. As a result the ratio Pennies:Nickels increases to 5:3. How many coins (pennies plus nickels) does Lenny have in his coin jar now?

(a) 600 coins

(b) 700 coins

(c) 800 coins.

(d) 900 coins

(e) None of these

Solution. **(c)** Let P and N be the respective number of pennies and nickels in Lenny's coin jar at the start. Then we know $P/N = 3/2 \Longrightarrow P = (3/2)N$. After adding Marty's 50 pennies, the ratio becomes $(P + 50)/N = 5/3 \Longrightarrow P + 50 = (5/3)N$. Substitute the first equation $P = (3/2)N$ into the second equation to get

$$\frac{3}{2}N + 50 = \frac{5}{3}N.$$

Solve for $N = 300$. Then $P = (3/2)300 = 450$. So the total coinage is

$$P + 50 + N = 450 + 50 + 300 = 800.$$

17. Two cubes (length = width = height) have respective volumes V_1 and V_2 that satisfy $V_1/V_2 = 10$. Let S_1 and S_2 be the respective surface areas of the cubes—so, S_1 corresponds to V_1 and S_2 corresponds to V_2. Determine the ratio of surface areas S_1/S_2.

(a) $S_1/S_2 = \sqrt[3]{150}$

(b) $S_1/S_2 = \sqrt[3]{10}$

(c) $S_1/S_2 = \sqrt[3]{200}$

(d) $S_1/S_2 = \sqrt[3]{100}$

(e) None of these

Solution. (d) Let x_1 and x_2 be the respective side lengths of the two cubes. Then, $V_1 = x_1^3$ and $V_2 = x_2^3$. By assumption, $V_1/V_2 = 10 \implies x_1^3/x_2^3 = 10 \implies x_1/x_2 = \sqrt[3]{10}$. Surface area is computed as follows, $S_1 = 6x_1^2$ and $S_2 = 6x_2^2$, and so

$$\begin{aligned} S_1/S_2 &= (6x_1^2)/(6x_2^2) \\ &= (x_1/x_2)^2 \\ &= (\sqrt[3]{10})^2 \\ &= \sqrt[3]{100}. \end{aligned}$$

18. If the pattern of digits 12345432123454321... is repeated indefinitely, what will be the 2014th digit?

(a) 1

(b) 2

(c) 3

(d) 4

(e) None of these

Solution. (d) The pattern is to repeat the 8-digit piece 12345432. We divide 8 into 2014 to get $2014 = 251 \times 8 + 6$, so the remainder is 6. This means the 2014th digit will be the 6th digit in 12345432, which happens to be 4.

19. The digit sum of a number is the sum of its decimal digits. For example, the digit sum of the number 3206 is $3 + 2 + 0 + 6 = 11$. Determine the digit sum of the number $(10^{2014} + 1)^4$.

(a) 10

(b) 12

(c) 14

(d) 16

(e) None of these

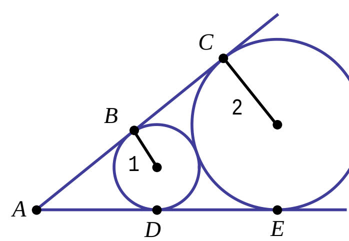
Solution. **(d)** First expand $(10^n+1)^4$,

$$\begin{aligned}(10^n+1)^4 &= (10^n+1)^2(10^n+1)^2\\ &= (10^{2n}+2\cdot 10^n+1)(10^{2n}+2\cdot 10^n+1)\\ &= 10^{4n}+4\cdot 10^{3n}+6\cdot 10^{2n}+4\cdot 10^n+1.\end{aligned}$$

From this, it should be apparent that the digit sum of $(10^n+1)^4$ for any value of $n>0$ is

$$1+4+6+4+1=16.$$

20. The two pictured circles, with respective radii 1 and 2, are mutually tangent to each other and tangent to the lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{AE}$ at the indicated points. What is the length AC?

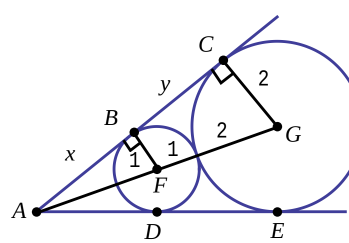


(a) $AC=5\sqrt{2}$ (b) $AC=2\sqrt{5}$

(c) $AC=2\sqrt{10}$ (d) $AC=\sqrt{11}$

(e) None of these

Solution. **(e)** Label the figure as pictured below. Also note that the circle radii are perpendicular to tangents.



By similar triangles $\triangle AFB \sim \triangle AGC$, we have $x/1 = (x+y)/2$ which implies $x = y$. That is, $AB = BC$ and so $AC = 2AB = 2x$. We may also conclude, for the same reason, that $AF = FG = 3$. The Pythagorean Theorem applied to $\triangle AFB$ gives $x^2+1 = AF^2 = 3^2$. We solve for $x = \sqrt{8} = 2\sqrt{2}$. Thus $AC = 2x = 4\sqrt{2}$, none of the answer choices provided.