CSU FRESNO MATHEMATICS FIELD DAY
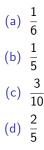
MAD HATTER MARATHON 9-10
PART II

April 26th, 2014

1. Consider the set of all solutions (x, y) to the equation $4x^2 = 4y^4 + 2014$, where x and y are integers. What is the sum of all y values occurring in such pairs (x, y)?

   (a) 0
   (b) 5
   (c) 36
   (d) 1296

2. You are playing a game on a game show that has the following rules. You are shown five prizes with different prices. You are given five price tags which give the prices of the prizes, and you must match the price tags to the prizes. Unfortunately, you have no idea how much any of the prizes costs, so you place the price tags randomly. What is the probability that you place exactly two price tags correctly?

(a) $\dfrac{1}{6}$

(b) $\dfrac{1}{5}$

(c) $\dfrac{3}{10}$

(d) $\dfrac{2}{5}$

3. The area of the largest triangle that can be inscribed in a semicircle of radius r is

   (a) $2r^2$
   (b) r^2
   (c) $\frac{1}{2}r^2$
   (d) $\frac{1}{4}r^2$

4. Suppose that five ordinary dice are rolled. What is the probability that at least one 6 appears?

(a) $\dfrac{5}{6}$

(b) $\left(\dfrac{5}{6}\right)^5$

(c) $1 - \left(\dfrac{5}{6}\right)^5$

(d) $\left(\dfrac{1}{6}\right)^5$

5. If the product of five integers is a multiple of 32, then what is the smallest number of these integers that must be even?

   (a) 5
   (b) 4
   (c) 2
   (d) 1

6. Simplify

$$\sqrt{9 + 3\sqrt{5}} - \sqrt{9 - 3\sqrt{5}}.$$

(a) $\sqrt{30}$
(b) 6
(c) $\sqrt{6}$
(d) 18

7. Simplify

$$\sqrt[4]{32} \div \sqrt[4]{2}.$$

(a) $\sqrt{2}$
(b) 2
(c) 4
(d) 8

8. If $a < 0$ and $b > 0$, which of the following must be true?

    (a) $a + b < 0$
    (b) $a^2 - b^2 > 0$
    (c) $ab < 0$
    (d) $a^2 < b$

9. Suppose that $p(x)$ is a polynomial of degree 2, has roots -2 and 4, and satisfies $p(0) = 24$. What is the coefficient of x in $p(x)$?

(a) 6
(b) 2
(c) -2
(d) -6

10. For which value(s) of h (if any) will the system of equations below have no solution?

$$8x - 3y = 9$$
$$24x + hy = 27.$$

(a) -9

(b) 9

(c) 0

(d) There is no such value of h.

11. Joe can paint a barn in 4 hours if he works alone. Steve has never timed himself painting a barn, so he doesn't know how fast he is. Working together, the two finish the job in 2.4 hours. How long does it take Steve to paint the barn by himself?

   (a) 6 hours
   (b) 5 hours
   (c) 3 hours
   (d) 1.6 hours

12. Find all possible ordered pairs (A, B) of digits for which the decimal number $7A8B$ is divisible by 45.

    (a) $(k, 0)$, where k is an arbitrary nonnegative integer
    (b) $(k, 5)$, where k is an arbitrary nonnegative integer
    (c) $(3, 0)$ and $(7, 5)$
    (d) $(5, 0)$

13. A gold bar is a rectangular solid measuring $2 \times 3 \times 4$. It is melted down, and three cubes of equal size are constructed from this mold. What is the length of a side of each cube?

(a) 8
(b) 6
(c) 4
(d) 2

14. A lamppost is 20 feet high. How many feet from the base of the post should a person who is 5 feet tall stand in order to cast an 8-foot shadow?

   (a) 32
   (b) 24
   (c) 16
   (d) 4

15. Find all real numbers x such that

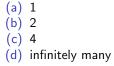$$\sqrt{4-x} + \sqrt{4+x} = 2x.$$

(a) $x = 0$

(b) $x = \pm\dfrac{\sqrt{15}}{2}$

(c) $x = 0$ and $x = \pm\dfrac{\sqrt{15}}{2}$

(d) $x = \dfrac{\sqrt{15}}{2}$

16. An aquarium has a rectanagular base measuring 100 cm by 40 cm and has a height of 50 cm. It is filled with water to a height of 40 cm. A brick measuring 40 cm by 20 cm by 10 cm is placed into the aquarium. By how many centimeters does the water rise?

(a) 2 cm
(b) 5 cm
(c) 8 cm
(d) 10 cm

17. The operation ∘ is defined by $x \circ y = 4x - 3y + xy$ for all real numbers x and y. For how many real numbers y does $3 \circ y = 12$.

    (a) 1
    (b) 2
    (c) 4
    (d) infinitely many

18. The diagonals of a rhombus measure 24 inches and 32 inches. The perimeter of the rhombus

   (a) must be 80 inches
   (b) must be 160 inches
   (c) must be $80\sqrt{2}$ inches
   (d) must be $160\sqrt{2}$ inches

19. A copying machine can make copies that are 80%, 100%, and 150% as large as the originals. By making copies of copies, what is the smallest number of times we must use the machine to get a copy of an original that is 324% as large as the original?

   (a) 5 times
   (b) 6 times
   (c) 8 times
   (d) It is impossible to make a copy that is 324% as large as the original.

20. During a certain period of days, it was observed that when it rained in the afternoon, it had been clear in the morning, and when it rained in the morning it was clear in the afternoon. It rained on 9 days, and it was clear on 6 afternoons and 7 mornings. Over what period of days was the weather observed?

(a) 13 days
(b) 12 days
(c) 11 days
(d) 10 days

21. A father and son have the same birthday, April 26. Today, the father becomes 42 and the son becomes 12. In what year will the father be exactly twice as old as the son?
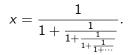
    (a) 2032
    (b) 2030
    (c) 2028
    (d) 2024

22. In $\triangle ABC$, $\angle C$ is three times as large as $\angle A$ and $\angle B$ is twice as large as $\angle A$. The ratio of AB to BC is
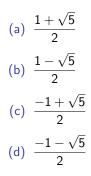
(a) $\dfrac{1}{2}$

(b) 2

(c) $\dfrac{1}{3}$

(d) $\dfrac{2}{3}$

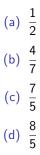23. The sum of the two roots of the equation $x^2 - 10x - 24 = 0$ is

    (a) -14
    (b) -10
    (c) 10
    (d) 14

24. Let x be the continued fraction defined by

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}.$$

What is the value of x?

(a) $\dfrac{1 + \sqrt{5}}{2}$

(b) $\dfrac{1 - \sqrt{5}}{2}$

(c) $\dfrac{-1 + \sqrt{5}}{2}$

(d) $\dfrac{-1 - \sqrt{5}}{2}$

25. The first three terms of a geometric sequence are the values x, y, and z, in that order. The first three terms of an arithmetic sequence are the values y, x, and z, in that order. If x, y, and z are distinct numbers, determine the ratio of the fifth term of the geometric sequence to the fifth term of the arithmetic sequence.

(a) $\dfrac{1}{2}$

(b) $\dfrac{4}{7}$

(c) $\dfrac{7}{5}$

(d) $\dfrac{8}{5}$

26. Two marks are made on an ordinary 12 inch ruler, one on each side of the ruler's midpoint. If one mark divides the ruler into two parts in the ratio 3:5, and the other mark divides the ruler into two parts in the ratio 5:11, how many inches separate these two marks?

(a) $2\frac{2}{3}$ inches

(b) $3\frac{3}{4}$ inches

(c) $4\frac{1}{5}$ inches

(d) $5\frac{1}{4}$ inches

27. John bought 3 apples and 2 oranges, paying $1.78. Then, he changed his mind and exchanged an orange for another apple, paying an additional 16¢. What is the price of a single orange?

   (a) 26¢
   (b) 38¢
   (c) 48¢
   (d) 52¢

28. A collection of twenty-five coins, whose total value is $2.75, consists of nickels, dimes, and quarters. If the nickels were dimes, the dimes were quarters, and the quarters were nickels, the total would be $3.75. How many quarters are there in the collection?

    (a) 3
    (b) 4
    (c) 5
    (d) 6

29. Two solutions of $(t + 2)^2 + 12t = 669$ are $t = 19$ and $t = -35$.
Which of the following is also a solution of this equation?

(a) -16
(b) 16
(c) 14
(d) None of the above

30. Which of the following statements are true?

   I. An acute triangle can be isosceles.
   II. An obtuse triangle can be isosceles.
   III. A right triangle can be isosceles.

   (a) I only
   (b) I and II
   (c) II only
   (d) I, II, and III

Solutions

1. A
2. A
3. B
4. C
5. D
6. C
7. B
8. C
9. A
10. D
11. A
12. C
13. D
14. B
15. D
16. A

17 D

18 A

19 B

20 C

21 A

22 B

23 C

24 C

25 D

26 B

27 A

28 C

29 D

30 D