MATH 149

Study Guide and Sample Problems for Test 1

Note: the actual test will consist of five questions, some of which will be computational, some will ask for a brief explanation, and some will require a rigorous detailed proof. Some of the problems will be very similar to homework problems and/or those discussed in class, but some will be different. So make sure that you understand well all the concepts discussed, know precise definitions and basic properties, rather than memorize how to solve specific problems.

1. Divisibility and congruences
   (a) State and prove divisibility tests for 2, 3, 4, 5, 8, 9, 10, and 11.
   (b) Prove that if $a|b$ and $a|c$, then $a|(b+c)$.
   (c) Prove that $a \equiv b$ $(mod\ 10)$ if and only if a and b have the same units digit.
   (d) Can a perfect square end with the digits 154?
2. Base 10
   (a) When 36 was appended to a natural number on the right, it increased by 103 times. What was the number?
   (b) A famous book of records once proclaimed that the largest known prime number is $23021^{377} - 1$. Prove that this cannot be correct.
   (c) Prove that if the units digit of a perfect square is 6, then its tens digit is odd.
3. Other bases
   (a) When a positive integer A is written in base 5, it has exactly the same digits (which are in the same order) as a positive integer B written in base 6. Which is larger, A or B?
   (b) An integer x has 3 digits when written in base 10. How many digits can it have when written in base 3?
   (c) Convert 2000_4 to base 7.
4. Combinatorics
   (a) Explain why there are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ways to choose k objects out of n.
   (b) Prove that $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$.
   (c) How many positive factors does 100000 have?
   (d) Explain why there are $\binom{n+k-1}{n}$ ways to distribute n identical balls into k distinguishable boxes.

(e) How many ways are there to distribute n identical balls into k distinguishable boxes if each box should contain at least one ball? At least two balls?

(f) How many ways are there to distribute n distinguishable balls into k distinguishable boxes?

(g) How many positive integer solutions does $x_1 + x_2 + x_3 + x_4 + x_5 = 10$ have?

5. Binomial theorem

   (a) Expand: $(x + y)^n$.

   (b) Find the first three terms in the expansions of $(2x + 1)^5$, $(x - 2y)^{10}$.

6. Probability, discrete and continuous variables.

   (a) There are 5 white, 6 red, and 7 blue balls in a bag. Two balls are drawn randomly. What is the probability that they are both blue?

   (b) How many ways are there to choose 3 cards from a deck of 52 cards? How many ways are there to choose 3 cards from the 12 "face" cards (J, Q, K)?

   (c) If three cards are chosen randomly for a deck of 52 cards, what is the probability that all three are face cards?

   (d) Two numbers, x and y, are randomly chosen in the interval $[0, 1]$. What is the probability that $2x + y > 1$?

   (e) A stick is broken at two random places. What is the probability that the longest piece is at least $\frac{3}{4}$ of the stick's length?