Finally, the set $\mathbf{C}$ of **complex numbers** is described in Section A.5. One method of construction is to use Kronecker's theorem, Theorem 4.3.8. An alternative is to consider ordered pairs of real numbers. Then addition and multiplication are defined as follows:

$$(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b) \cdot (c, d) = (ac - bd, ad + bc) .$$

The ordered pair $(a, b)$ is usually written $a + bi$, where $i^2 = -1$.

Detailed proofs of the assertions in this section can be found in various text books such as those by Landau and by Cohen and Ehrlich. The construction of the real numbers from the rationals is usually viewed as a part of analysis rather than algebra.

## A.3 Basic Properties of the Integers

We assume that the reader is familiar with the arithmetic and order properties of the integers, and indeed, we have freely used these properties throughout the book. In the interest of completeness we now explicitly list these properties, as well as their names.

**A.3.1 (Properties of Addition).**

**(a)** Closure: *Given any two integers* $a$ *and* $b$, *there is a unique integer* $a + b$.

**(b)** Associativity: *Given integers* $a, b, c$, *we have* $(a + b) + c = a + (b + c)$.

**(c)** Commutativity: *Given integers* $a, b$, *we have* $a + b = b + a$.

**(d)** Zero element: *There exists a unique integer* $0$ *such that* $a + 0 = a$ *for any integer* $a$.

**(e)** Inverses: *Given an integer* $a$, *there exists a unique integer, denoted by* $-a$, *such that* $a + (-a) = 0$.

**A.3.2 (Properties of Multiplication).**

**(a)** Closure: *Given any two integers* $a$ *and* $b$, *there is a unique integer* $a \cdot b = ab$.

**(b)** Associativity: *Given integers* $a, b, c$, *we have* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

**(c)** Commutativity: *Given integers* $a, b$, *we have* $a \cdot b = b \cdot a$.

**(d)** Identity element: *There exists a unique integer* $1 (\neq 0)$ *such that* $a \cdot 1 = a$ *for any integer* $a$.

**A.3.3 (Joint Property of Addition and Multiplication).**

Distributivity: *Given integers* $a, b, c$, *we have* $a(b + c) = ab + ac$.

**A.3.4 (Properties of Order).** *There exists a subset* $\mathbf{Z}^+ \subset \mathbf{Z}$, *called the* set of positive integers, *which satisfies the following properties:*

**(a)** Closure under addition: *If* $a, b \in \mathbf{Z}^+$, *then* $a + b \in \mathbf{Z}^+$.

**(b)** Closure under multiplication: *If* $a, b \in \mathbf{Z}^+$, *then* $ab \in \mathbf{Z}^+$.

**(c)** Trichotomy: *Given* $a \in \mathbf{Z}$, *exactly one of the following holds:*

(i) $a \in \mathbf{Z}^+$ , (ii) $a = 0$ , (iii) $-a \in \mathbf{Z}^+$ .

A number of these properties are redundant. Our purpose is to provide a working knowledge of the system of integers, and so we have not given the most economical list of properties. Rather than investigating the foundations of the number systems, we will be content with simply making the following statement: Together with the well-ordering principle, the above list of thirteen properties completely characterizes the set of integers.

The following proposition lists some of the usual arithmetic properties of the set of integers. These properties hold in a more general setting, which is studied in Chapter 5. We will use the notation $a - b$ for $a + (-b)$.

**A.3.5 Proposition.** *Let* $a, b, c \in \mathbb{Z}$.
**(a)** *If* $a + b = a + c$, *then* $b = c$.
**(b)** $-(-a) = a$.
**(c)** $a \cdot 0 = 0$.
**(d)** $(-a)(-b) = ab$.

We introduce the usual order symbols as follows. We say that $a$ is greater than $b$, denoted by $a > b$, if $a - b \in \mathbb{Z}^+$. For $a > b$ we also write $b < a$ (read "$b$ is less than $a$"), and $a \geq b$ (read "$a$ is greater than or equal to $b$") denotes that $a = b$ or $a > b$. Finally, the absolute value of $a$, denoted by $|a|$, is equal to $a$ if $a \in \mathbb{Z}^+$ or $a = 0$ and is equal to $-a$ if $-a \in \mathbb{Z}^+$. The proof of the next proposition is left as an exercise.

**A.3.6 Proposition.** *Let* $a, b, c \in \mathbb{Z}$.
**(a)** *If* $a > 0$, *then* $a \geq 1$.
**(b)** *If* $a > b$ *and* $b > c$, *then* $a > c$.
**(c)** *If* $a > b$, *then* $a + c > b + c$.
**(d)** $a < 0$ *if and only if* $-a \in \mathbb{Z}^+$.
**(e)** *If* $a > b$ *and* $c > 0$, *then* $ac > bc$.
**(f)** *If* $a > b$ *and* $c < 0$, *then* $ac < bc$.
**(g)** $|a| \geq 0$, *and* $|a| = 0$ *if and only if* $a = 0$.
**(h)** *If* $a > 0$, *then* $|b| \leq a$ *if and only if* $-a \leq b \leq a$.
**(k)** $|ab| = |a||b|$.
**(m)** $|a + b| \leq |a| + |b|$.
**(n)** *If* $ab = ac$ *and* $a \neq 0$, *then* $b = c$.

## A.4 Induction

If one develops the natural numbers from the Peano postulates, then mathematical induction is taken to be one of the postulates. On the other hand, if one uses the list of properties given in Section A.3 as a starting point, then the well-ordering