MATH 149

Study Guide and Sample Problems for Test 2

Note: the actual test will consist of five questions, some of which will be computational, some will ask for a brief explanation, and some will require a rigorous detailed proof. Some of the problems will be very similar to homework problems and/or those discussed in class, but some will be different. So make sure that you understand well all the concepts discussed, know precise definitions and basic properties, rather than memorize how to solve specific problems.

1. Probability, discrete and continuous variables.
   (a) There are 5 white, 6 red, and 7 blue balls in a bag. Two balls are drawn randomly. What is the probability that they are both blue?
   (b) How many ways are there to choose 3 cards from a deck of 52 cards? How many ways are there to choose 3 cards from the 12 "face" cards (J, Q, K)?
   (c) If three cards are chosen randomly for a deck of 52 cards, what is the probability that all three are face cards?
   (d) Two numbers, x and y, are randomly chosen in the interval $[0, 1]$. What is the probability that $2x + y > 1$?
   (e) A stick is broken at two random places. What is the probability that the longest piece is at least $\frac{3}{4}$ of the stick's length?
2. Formal construction of number systems
   (a) Give the definition of the set of natural numbers (with Peano postulates)
   (b) Give the definition of the set of integer numbers as equivalence classes of pairs of natural numbers, and definitions of addition and multiplication.
   (c) Give the definition of the set of rational numbers as equivalence classes of pairs of integer numbers, and definitions of addition and multiplication.
   (d) Give the definition of the set of real numbers as equivalence classes of Cauchy sequences of rational numbers, and definitions of addition and multiplication.
   (e) Give the definition of the set of complex numbers either as formal expressions of the form $a + bi$, and definitions of addition and multiplication.
3. Rational and irrational numbers.
   (a) Prove that the quotient of two nonzero rational numbers is rational.
   (b) Prove that the following statement is false: "the quotient of two nonzero irrational numbers is irrational".
   (c) Prove that $\sqrt{5}$ is irrational.
   (d) Convert the following decimals into fractions: 12.345, 12.3(45).

(e) What is the 149th digit after the decimal point in the decimal representation of $\frac{1}{7}$?

(f) Explain why a decimal represents a rational number if and only if it is either terminating or periodic.

(g) Show that $\sqrt{2}$ cannot be written in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Q}$ (i.e. $\sqrt{2} \notin \mathbb{Q}(\sqrt{3})$).

4. Complex numbers.

(a) Find the additive and multiplicative inverses of the element $4 + 3i$ in $\mathbb{C}$.

(b) Compute: i^{149}, $(\sqrt{3} + i)^9$.

(c) Find at least one complex solution to $x^4 = -1$.

(d) Find the sum of the infinite series $1 + \frac{1}{2}i + \frac{1}{4} + \frac{1}{8}i + \frac{1}{16} + \frac{1}{32}i + \ldots$.

5. Groups, abelian groups, rings, commutative rings, fields.

(a) Give the definition of a group; abelian group. Give an example of a non-ableian group.

(b) Give an example of a ring that is not a field.

(c) Which of the following sets with the usual addition and multiplication are fields: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}^+$ (the set of positive real numbers), $\mathbb{R}$? For those the are not fields, say which axioms do not hold.

(d) Give an example of a set of familiar objects (familiar to high school students) with an operation of multiplication for which the following property does not hold: if $ab = 0$, then $a = 0$ or $b = 0$.