MATH 149

Study Guide and Sample Problems for Test 3

Note: the actual test will consist of five questions, some of which will be computational, some will ask for a brief explanation, and some will require a rigorous detailed proof. Some of the problems will be very similar to homework problems and/or those discussed in class, but some will be different. So make sure that you understand well all the concepts discussed, know precise definitions and basic properties, rather than memorize how to solve specific problems.

1. Functions
   (a) What is a function? List at least 4 ways to define/represent a function.
   (b) Give the definitions of: domain, codomain, range, intercepts, roots, graph.
   (c) Determine which of the following statements are true, which are false, and which do not make sense or are incorrectly worded. For the last group, explain what exactly is incorrect, and, if possible, propose a correction.
      i. The point $(2, 3)$ satisfies the function $f(x) = x^2 - 1$.
      ii. If the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \frac{1}{x}$, then the value $f(0)$ does not exist.
      iii. The point $(3, 4)$ lies on the graph of the function $x^2 + y^2 = 25$.
      iv. The graph of the equation $y = x + 5$ has two intercepts.
      v. One way to find all horizontal intercepts of the function $f(x)$ is to solve the equation $f(x) = 0$.
      vi. The graph of a function $f : \mathbb{R} \to \mathbb{R}$ can have any number of vertical asymptotes, from 0 to infinitely many.
      vii. The graph of a function $f : \mathbb{R} \to \mathbb{R}$ can have any number of horizontal asymptotes, from 0 to infinitely many.
      viii. A function $f : \mathbb{R} \to \mathbb{R}$ can have any number of roots, from 0 to infinitely many.
      ix. The graph of a function $f : \mathbb{R} \to \mathbb{R}$ can have any number of x-intercepts, from 0 to infinitely many.
      x. The graph of a function $f : \mathbb{R} \to \mathbb{R}$ can have any number of y-intercepts, from 0 to infinitely many.
2. Polynomials, polynomial equations and inequalities.
   (a) Give the definition of a polynomial.
   (b) Derive the quadratic formula.
   (c) Prove that the vertex of the parabola given by $y = ax^2 + bx + c$ is located at $\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$.

(d) Sketch a graph of $f(x) = (x-4)^2x(x+1)^3$.

(e) What is the largest value of c for which the equation $4x^2 + 6x + c = 0$ has a real root?

(f) State Vieta's formulas (and use them to solve problems).

(g) Solve for x:

i. $(x-4)^2x(x+1)^3 > 0$

ii. $(x-4)^2x(x+1)^3 \geq 0$

iii. $(x-4)^2x(x+1)^3 \leq 0$

(h) Al is thinking of a function, $f(x)$. He reveals to Bob that the function is a polynomial of the form $f(x) = ax^8 + bx^4 + cx^2 + dx + e$, where a, b, c, d, and e are real coefficients. Bob wishes to determine the value of e. For any real number x that Bob asks about, Al will tell him the value of $f(x)$. At least how many values of x must Bob ask about in order to definitively determine the value of e?

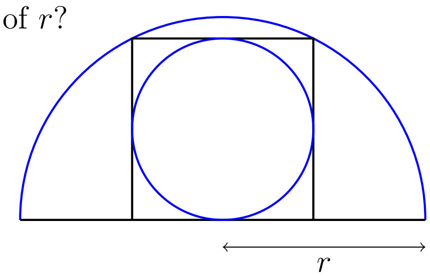
3. Pythagorean Theorem.

(a) Give at least two proofs of the Pythagorean Theorem.

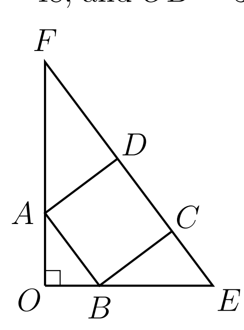
(b) Suppose that 12, 13, and 15 are the lengths (possibly not in this order) of two sides of an acute triangle and of the height over the third side of triangle. Find the area of the triangle.

(c) In regular hexagon $ABCDEF$ with side length 1, AD intersects BF at G. Find the length of CG.

(d) The semicircle pictured has a radius equal to r inches. The square is inscribed in the semicircle and the smaller circle is inscribed in the square. What is the area of the smaller circle in terms of r?



(e) If $ABCD$ is a square, OEF is a right triangle, $OA = 48$, and $OB = 36$, what is the length of the segment EF?



4. Coordinate Geometry.

   (a) Prove the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

   (b) Describe the set of all points (x, y) that satisfy the following equation:

      i. $x^2 + y^2 - 4x + 6y + 5 = 0$

      ii. $x^2 + y^2 - 4x + 6y + 13 = 0$

      iii. $x^2 + y^2 - 4x + 6y + 20 = 0$

   (c) Find all values of m for which the curves $x^2 + y^2 = 1$ and $y = x^2 + m$ have exactly one point in common.

5. Similarity.

   (a) Alan was calculating the volume of a sphere, but in the calculation he mistakenly used the value of the diameter instead of the radius of the sphere. What should he do with his result to get the correct answer?

   (b) Two cubes have respective volumes V_1 and V_2 that satisfy $V_1/V_2 = 10$. Let S_1 and S_2 be the respective surface areas of the cubes, so, S_1 corresponds to V_1 and S_2 corresponds to V_2. Determine the ratio of surface areas S_1/S_2.

   (c) The ratio of the radius of the first circle to that of the second circle is $5 : 2$, and the ratio of the area of the second circle to that of the third circle is $9 : 25$. What is the ratio of the circumference of the third circle to that of the first circle?

   (d) Write your own problem involving similar figures and their perimeters, areas, or volumes.