Math 150 **Maria Nogin**

Curve Sketching

$$\text{Example. } f(x) = \frac{2x^2}{x^2-1}$$

A. Domain. $f(x)$ is undefined when $x^2 - 1 = 0 \Leftrightarrow (x-1)(x+1) = 0 \Leftrightarrow x = 1, \ -1$.
Thus Domain$(f) = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$.

B. Intercepts. y-intercept: $y(0) = 0$.
x-intercepts: $f(x) = 0 \Leftrightarrow \dfrac{2x^2}{x^2-1} = 0 \Leftrightarrow x = 0$.
Thus $(0, 0)$ is the only intercept.

C. Symmetry. $\dfrac{2(-x)^2}{(-x)^2-1} = \dfrac{2x^2}{x^2-1} = 0$
i.e. $f(-x) = f(x)$, i.e. $f(x)$ is even, therefore the graph is symmetric about the y-axis.

D. Asymptotes.

horizontal: $$\lim_{x\to+\infty} \frac{2x^2}{x^2-1} = \lim_{x\to+\infty} \frac{2}{1-\frac{1}{x^2}} = 2$$
$$\lim_{x\to-\infty} \frac{2x^2}{x^2-1} = \lim_{x\to-\infty} \frac{2}{1-\frac{1}{x^2}} = 2$$
$\Rightarrow$ $y = 2$ is a horizontal asymptote.

vertical: $$\lim_{x\to1^+} \frac{2x^2}{x^2-1} \left(= \frac{2}{+0}\right) = +\infty$$
$$\lim_{x\to1^-} \frac{2x^2}{x^2-1} \left(= \frac{2}{-0}\right) = -\infty$$
$$\lim_{x\to-1^+} \frac{2x^2}{x^2-1} \left(= \frac{2}{-0}\right) = -\infty$$
$$\lim_{x\to-1^-} \frac{2x^2}{x^2-1} \left(= \frac{2}{+0}\right) = +\infty$$
$\Rightarrow$ $x = 1$ and $x = -1$ are vertical asymptotes.

E. Increase / decrease.
$$f'(x) = \frac{4x(x^2-1) - 2x^2 \cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$
since $(x^2-1)^2 > 0$ for $x \neq \pm 1$,
$f'(x) > 0$ when $x < 0 \Rightarrow f(x)$ is increasing on $(-\infty, -1)$ and on $(-1, 0)$.
$f'(x) < 0$ when $x > 0 \Rightarrow f(x)$ is decreasing on $(0, 1)$ and on $(1, +\infty)$.

F. Max / min.
$$f'(x) = 0 \Rightarrow \frac{2x^2}{x^2-1} = 0 \Rightarrow x = 0.$$
At 0, $f'(x)$ changes from $+$ to $-$, so 0 is a local maximum.
$f'(x)$ does not exist when $x^2 - 1 = 0 \Rightarrow x = \pm 1$, but 1 and -1 are not in the domain of $f(x)$, so they are not critical numbers.

G. Concavity and inflection points.

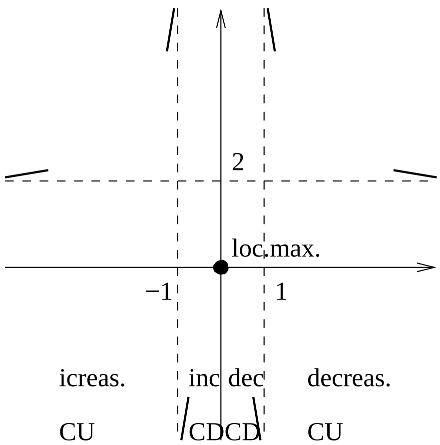
$$f''(x) = \frac{-4(x^2-1)^2 - (-4x)2(x^2-1)2x}{(x^2-1)^4} = \frac{(x^2-1)[-4(x^2-1) + 4x \cdot 2 \cdot 2x]}{(x^2-1)^4} =$$

$$= \frac{-4x^2 + 4 + 16x^2}{(x^2-1)^3} = \frac{12x^2+4}{(x^2-1)^3}.$$

since $12x^2 + 4 > 0$ for all x,
$f''(x) > 0$ when $x^2 - 1 > 0 \Rightarrow f(x)$ is CU on $(-\infty, -1)$ and on $(1, +\infty)$.
$f''(x) < 0$ when $x^2 - 1 < 0 \Rightarrow f(x)$ is CD on $(-1, 1)$.
$f(x)$ changes the direction of concavity at ± 1, but 1 and -1 are not in the domain of $f(x)$, so there are no inflection points.

H. Graph.

first sketch the asymptotes (also notice that when x is large or large negative, $f(x) = \dfrac{2x^2}{x^2-1} = \dfrac{2}{1-\frac{1}{x^2}}$ is slightly bigger than 2 because $1 - \dfrac{1}{x^2}$ is slightly smaller than 1, so the graph approaches the horizontal asymptote from above), plot the intercepts (only $(0,0)$ in our case), maxima ($(0,0)$ again), minima (none), and inflection points (none), and indicate the intevals of increase, decrease, and concavity:



now sketch the graph: