Math 151

Solutions to selected homework problems

Section 2.1, Problem 2(ab):

In each of the following parts, determine whether the given function is one-to-one and whether it is onto.

(a) $f : \mathbb{R} \to \mathbb{R} : \ f(x) = x^2$

(c) $f : \mathbb{R}^+ \to \mathbb{R}^+ : \ f(x) = x^2$

Solution:

(a) The function f is not one-to-one because e.g. $f(1) = f(-1)$ while $1 \neq -1$. It is not onto because e.g. -1 is not in the image (there is no real number x such that $x^2 = -1$).

(c) The function f is one-to-one because $f(x_1) = f(x_2)$, i.e. $x_1^2 = x_2^2$ implies $x_1 = \pm x_2$. Since $x_1, x_2 > 0$, $x_1 = x_2$. It is onto because for each $y \in \mathbb{R}^+$ there exists an $x \in \mathbb{R}^+$ such that $f(x) = y$, e.g. $x = \sqrt{y}$.

Section 2.1, Problem 14:

Let $f : A \to B$ and $g : B \to C$ be one-to-one and onto. Show that $(g \circ f)^{-1}$ exists and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Solution:

Since both f and g are one-to-one and onto, $g \circ f : A \to C$ is one-to-one and onto. Therefore it has a unique inverse. To show that $f^{-1} \circ g^{-1}$ is its inverse, we have to check that $(f^{-1} \circ g^{-1}) \circ (g \circ f) = 1_A$ and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = 1_C$. Indeed, since composition of functions is associative, we have
$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ 1_B \circ f = f^{-1} \circ f = 1_A$ and
$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ 1_B \circ g^{-1} = g \circ g^{-1} = 1_C$.

Section 2.2, Problem 1(b):

It is shown in Theorem 2.2.7 that if $f : S \to T$ is a function, then there is a one-to-one correspondence between the elements of $f(S)$ and the equivalence classes of S/f. For each of the following functions, find $f(S)$ and S/f and exhibit the one-to-one correspondence between them.

(b) $g : \mathbb{Z} \to \mathbb{Z}_{12}$ given by $g(n) = [8n]_{12}$ for all $n \in \mathbb{Z}$.

Solution:

First we calculate a few images: $g(0) = [0]_{12}$, $g(1) = [8]_{12}$, $g(2) = [4]_{12}$, $g(3) = [0]_{12}$, $g(4) = [8]_{12}$. It appears that the images start repeating, and two integers have the same image if and only if they are congruent modulo 3.

Proof of the above statement: $g(x_1) = g(x_2)$ iff $[8x_1]_{12} = [8x_2]_{12}$ iff $8x_1 \equiv 8x_2 \pmod{12}$ iff $2x_1 \equiv 2x_2 \pmod{12}$ iff $x_1 \equiv x_2 \pmod{12}$.

Thus the set of images is $g(\mathbb{Z}) = \{[0]_{12}, [4]_{12}, [8]_{12}\}$, the set of equivalence classes determined by g is $\mathbb{Z}/g = \{[0]_3, [1]_3, [2]_3\}$, and the one-to-one correspondence induced by g is $\overline{g} : \mathbb{Z}/g \to g(\mathbb{Z})$ given by $\overline{g}([0]_3) = [0]_{12}$, $\overline{g}([1]_3) = [8]_{12}$, $\overline{g}([2]_3) = [4]_{12}$.

Section 2.2, Problem 4:

Let S be the set of all ordered pairs (m, n) of positive integers. For $(a_1, a_2) \in \S$ and $(b_1, b_2) \in \S$, define $(a_1, a_2) \sim (b_1, b_2)$ if $a_1 + b_2 = a_2 + b_1$. Show that $\sim$ is an equivalence relation.

Solution:

For any a_1 and a_2, since $a_1 + a_2 = a_2 + a_1$, $(a_1, a_2) \sim (a_1, a_2)$. Thus $\sim$ is reflexive.

If $(a_1, a_2) \sim (b_1, b_2)$, then $a_1 + b_2 = a_2 + b_1$, then $b_1 + a_2 = b_2 + a_1$, i.e. $(b_1, b_2) \sim (a_1, a_2)$. Thus $\sim$ is symmetric.

If $(a_1, a_2) \sim (b_1, b_2)$ and $(b_1, b_2) \sim (c_1, c_2)$, then $a_1 + b_2 = a_2 + b_1$ and $b_1 + c_2 = b_2 + c_1$. Then $a_1 - a_2 = b_1 - b_2$ and $b_1 - b_2 = c_1 - c_2$. It follows that $a_1 - a_2 = c_1 - c_2$, then $a_1 + c_2 = a_2 + c_1$, i.e. $(a_1, a_2) \sim (c_1, c_2)$. Thus $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation.

Section 2.2, Problem 12(a,b):

Done in class on 9/26.