Math 151

Solutions to selected homework problems

Matrices, Problem 8:

Show that the product of two upper triangular matrices is upper triangular.

Solution:

Let A and B be $n \times n$ upper triangular matrices, and let $C = AB$. We want to show that C is upper triangular, i.e. that if $i > j$, then $c_{ij} = 0$. We will prove this by contrapositive, i.e. we will prove that if $c_{ij} \neq 0$, then $i \leq j$.

Assume $c_{ij} \neq 0$. Since $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, there exists k such that $a_{ik} b_{kj} \neq 0$. It follows that $a_{ik} \neq 0$ and $b_{kj} \neq 0$. Since A and B are upper triangular, $i \leq k$ and $k \leq j$. Therefore $i \leq j$.

Section 3.1, Problem 1:

Using ordinary addition of integers as the operation, show that the set of even integers is a group, but the set of odd integers is not.

Solution:

Let us first consider the set of even integers. The ordinary addition is indeed a binary operation on this set since the sum of two even integers is even. It is associative because, as we know, addition of any integers is associative. The number 0 is even, and is the additive identity. Finally, the additive inverse of any even integer is even, so is contained in the set. Thus we have a group.

For the set of odd integers, the ordinary addition is not a binary operation, becase the sum of two odd integers is even (and not odd). Therefore it is not a group.

Section 3.1, Problem 2(bf):

For each binary operation $*$ defined on a set below, determine whether or not $*$ gives a group structure on the set. If it is not a group, say which axioms fail to hold.

(b) Define $*$ on $\mathbb{Z}$ by $a * b = max\{a, b\}$.

Solution:

This is a binary operation since the maximum of two integers is an integer.

Associativity holds: $max\{max\{a, b\}, c\} = max\{a, b, c\} = max\{a, max\{b, c\}\}$.

However, there is no identity element. We will prove this by contradiction. Suppose there is an identity element $e \in \mathbb{Z}$, so for any $a \in \mathbb{Z}$, $max\{a, e\} = a$. Consider $a = e - 1$. Then $max\{a, e\} = max\{e - 1, e\} = e \neq a$. We have a contradiction.

Since there is no identity, it does not make sense to talk about the inverse.

So this is not a group.

(f) Define $*$ on $\mathbb{Q}$ by $a * b = ab$.

Solution:

This is a binary operation since the product of two rational numbers is rational.

Associativity holds, since, as we know, multiplication of any real numbers is associative.

There is an identity element, namely, 1: for any rational number a, $a \cdot 1 = 1 \cdot a = a$.

However, 0 does not have an inverse: there is no rational number a such that $a \cdot 0 = 1$.

So this is not a group.

Section 3.1, Problem 10:

Show that the set $A = \{f_{m,b} : \mathbb{R} \to \mathbb{R} \mid m \neq 0 \text{ and } f_{m,b}(x) = mx + b\}$ of affine functions from $\mathbb{R}$ to $\mathbb{R}$ forms a group under composition of functions.

Solution:

First we will show that composition is a binary operation on this set: let $f_{m,b}(x) = mx+b$ and $f_{n,c}(x) = nx + c$, where $m \neq 0$, $n \neq 0$. then $f_{m,b} \circ f_{n,c}(x) = f_{m,b}(nx + c) = m(nx + c) + b = (mn)x + (mc + b)$. Since $mn \neq 0$, $f_{m,b} \circ f_{n,c} \in A$.

Associativity holds, since, as we know, composition of any functions is associative.

There is an identity element, namely, $f_{1,0}$ ($f_{1,0}(x) = x$): it is easy to check that for any $f_{m,b}$, $f_{m,b} \circ f_{1,0} = f_{1,0} \circ f_{m,b} = f_{m,b}$.

Finally, for any $f_{m,b}$, the function $f_{\frac{1}{m}, -\frac{b}{m}}$ is its inverse:
$f_{\frac{1}{m}, -\frac{b}{m}} \circ f_{m,b}(x) = f_{\frac{1}{m}, -\frac{b}{m}}(mx + b) = \frac{1}{m}(mx + b) - \frac{b}{m} = x$, and similarly it can be checked that $f_{m,b} \circ f_{\frac{1}{m}, -\frac{b}{m}}(x) = x$.