Math 151

Solutions to selected homework problems

Section 3.6, Problem 1(b,d):

Find the orders of each of these permutations.

(b) $(1, 2, 5)(2, 3, 4)(5, 6)$

(d) $(1, 2, 3)(2, 4, 3, 5)(1, 3, 2)$

Solution:

We know that if a permutation is written as a product of disjoint cycles, then its order is the LCM of the lengths of the cycles. So first we write each permutation as a product of disjoint cycles.

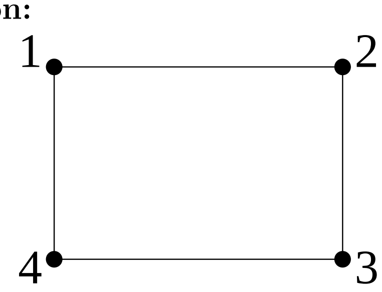
(b) $(1, 2, 5)(2, 3, 4)(5, 6) = (1, 2, 3, 4, 5, 6)$. The order of this permutation is 6.

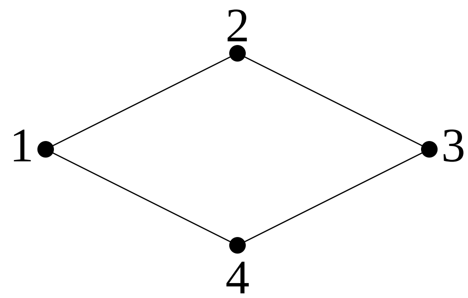
(d) $(1, 2, 3)(2, 4, 3, 5)(1, 3, 2) = (1, 5, 3, 4)(2) = (1, 5, 3, 4)$. The order of this permutation is 4.

Section 3.6, Problem 4:

Find the permutations that correspond to the rigid motions of a rectangle that is not a square. Do the same for the rigid motions of a rhombus (diamond) that is not a square.

Solution:





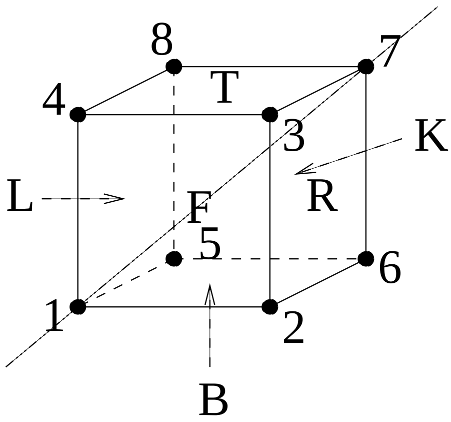
Rigid motions of the rectangle: (1), $(1, 4)(2, 3)$, $(1, 2)(3, 4)$, $(1, 3)(2, 4)$.

Rigid motions of the rhombus: (1), $(1, 3)$, $(2, 4)$, $(1, 3)(2, 4)$.

Section 3.6, Problem 9:

A rigid motion of a cube can be thought of either as a permutation of its eight vertices or as a permutation of its six sides. Find a rigid motion of the cube that has order 3, and express the permutation that represents it in both ways, as a permutation on eight elements and as a permutation on six elements.

Solution:



Rigid motion of order 3: rotation about the line shown (passing through vertices 1 and 7), through an angle of 120 degrees.

Permutation of vertices: $(2,4,5)(3,8,6)$.

Permutation of sides: (F,L,B)(T,K,R) (letters F, L, B, T, K, and R stand for front, left, bottom, top, back, and right respectively).

Section 3.6, Problem 13:

List the elements of A_4.

Solution:

Recall that A_4 consists of all even permutations in S_4.

Elements of A_4 are: (1), $(1,2,3)$, $(1,3,2)$, $(1,2,4)$, $(1,4,2)$, $(1,3,4)$, $(1,4,3)$, $(2,3,4)$, $(2,4,3)$, $(1,2)(3,4)$, $(1,3)(2,4)$, $(1,4)(2,3)$.

(Just checking: the order of a subgroup must divide the order of the group. We have listed 12 elements, $|S_4| = 24$, and $12 \mid 24$.)

Section 3.6, Problem 25:

Show that S_n is isomorphic to a subgroup of A_{n+2}.

Solution:

Hint: define $\phi : S_n \to A_{n+2}$ by

$$\phi(\sigma) = \begin{cases} \sigma & \text{if } \sigma \text{ is even,} \\ (\sigma)(n+1, n+2) & \text{if } \sigma \text{ is odd} \end{cases}$$

Show that ϕ is a homomorphism.

Section 3.7, Problem 3(b):

Show that the following functions are homomorphisms.

(b) $\phi : \mathbb{R}^\times \to \mathbb{R}^\times$ defined by $\phi(x) = \frac{x}{|x|}$.

Solution:

$\phi(xy) = \frac{xy}{|xy|} = \frac{xy}{|x| \cdot |y|} = \frac{x}{|x|} \frac{y}{|y|} = \phi(x)\phi(y)$.

Section 3.7, Problem 4:

Let G be an abelian group, and let n be any positive integer. Show that the function $\phi : G \to G$ defined by $\phi(x) = x^n$ is a homomorphism.

Solution:

Since the group is abelian, any two elements x and y commute. Thus $\phi(xy) = (xy)^n = (xy)(xy)\ldots(xy) = xx\ldots xyy\ldots y = x^n y^n$.

Section 3.7, Problem 7(b,d):

Which of the following functions are homomorphisms?

(b) $\phi : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ defined by $\phi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

(d) $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab$

Solution:

(b) $\phi(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \phi(a)\phi(b)$, so ϕ is a homomorphism.

(d) Since $\phi(I_2) = 0 \neq 1$, the function is not a homomorphism (recall that any homomorphism must send the identity element to the identity element).

Section 3.7, Problem 9:

Let ϕ be a group homomorphism of G_1 onto G_2. Prove that if G_1 is abelian then so is G_2; prove that if G_1 is cyclic then so is G_2. In each case, give a counterexample to the converse of the statement.

Solution:

Let G_1 be abelian. Let $a, b \in G_2$. Since ϕ is onto, there exist $x, y \in G_1$ such that $\phi(x) = a$ and $\phi(y) = b$. Then $ab = \phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x) = ba$, so G_2 is abelian.

Let G_1 be cyclic. Then $G_1 =< x >$ for some $x \in G_1$, i.e. $G_1 = \{x^n \mid n \in \mathbb{Z}\}$. Then $G_2 = \phi(G_1) = \{\phi(x^n) \mid n \in \mathbb{Z}\} = \{(\phi(x))^n \mid n \in \mathbb{Z}\} =< \phi(x) >$, so G_2 is cyclic.

Now consider $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ defined by $\phi(A) = \det(A)$. Since $GL_2(\mathbb{R})$ is not abelian but $\mathbb{R}^\times$ is abelian, we have a counterexample to the converse of the first statement.

Finally, consider $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ definded by $\phi((x, y)) = (x, 0)$. Since $\mathbb{Z}_2$ is cyclic, but $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not, we have a counterexample to the converse of the second statement.

Alternatively, $\phi : S_3 \to \mathbb{Z}_2$ defined by

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even,} \\ 1 & \text{if } \sigma \text{ is odd} \end{cases}$$

is a counterexample to the converses of both statements, since Z_2 is both abelian and cyclic, but S_3 is neither.

(Note: there are many other counterexamples.)