Math 151

Solutions to selected homework problems

Section 3.7, Problem 5:

Let G be the multiplicative group $\mathbb{Z}_{15}^{\times} = \{1, 2, 4, 7, 8, 11, 13, 14\}$, and let $n = 2$. Compute the values of the function defined in Exercise 4 ($\phi : G \to G$, $\phi(x) = x^n$), and find its kernel and the image of G.

Solution:

The values are: $\phi(1) = 1$, $\phi(2) = 4$, $\phi(4) = 16 = 1$, $\phi(7) = 49 = 1$, $\phi(8) = 64 = 4$, $\phi(11) = 121 = 1$, $\phi(13) = 169 = 4$, $\phi(14) = 196 = 4$.

So $\ker(\phi) = \{1, 4, 7, 11\}$ and $\phi(G) = \{1, 4\}$.

Section 3.7, Problem 15:

Prove that the intersection of two normal subgroups is a normal subgroup.

Solution:

Let H and K be normal subgroups of G. Let $x \in H \cap K$. Then $x \in H$ and $x \in K$. For any element $g \in G$, $gxg^{-1} \in H$ (since H is normal) and $gxg^{-1} \in K$ (since K is normal). So $gxg^{-1} \in H \cap K$. Thus $H \cap K$ is normal.

Section 3.8, Problem 7:

Let H be a subgroup of G, and let $a \in G$. Show that aHa^{-1} is a subgroup of G that is isomorphic to H.

Solution:

We have proved earlier (section 3.4, problem 15) that $\phi : G \to G$ defined by $\phi(x) = axa^{-1}$ is a group isomorphism. Therefore the image of H, aHa^{-1}, is a subgroup of G. Moreover, since ϕ is one-to-one as a function from G to G, it is one-to-one as a function from H to aHa^{-1}. Also, $\phi : H \to aHa^{-1}$ is onto (any element of aHa^{-1} has the form aha^{-1} for some $h \in H$, and $\phi(h) = aha^{-1}$) and preserves the operation (because it preserves the operation as a function from G to G).

Section 3.8, Problem 8:

Let H be a subgroup of G. Show that H is normal in G if and only if $aHa^{-1} = H$ for all $a \in G$.

Solution:

By prop. 3.8.8, H is normal iff $aH = Ha$ for all $a \in G$. Since $aH = Ha$ is equivalent to $aHa^{-1} = H$, we have the desired statement.

Section 4.1, Problem 9:

Let a be a nonzero element of a field F. Show that $(a^{-1})^{-1} = a$ and $(-a)^{-1} = -a^{-1}$.

Solution:

For the first statement, we have:

$(a^{-1})^{-1} = (a^{-1})^{-1} \cdot 1 = (a^{-1})^{-1}(a^{-1}a) = ((a^{-1})^{-1}a^{-1})a = 1 \cdot a = a.$

Also, by prop. 4.1.3 part (e),

$(-a)^{-1} = (-a)^{-1} \cdot 1 = (-a)^{-1}(a \cdot a^{-1}) = (-a)^{-1}(-a)(-a^{-1}) = ((-a)^{-1}(-a))(-a^{-1}) = 1 \cdot (-a^{-1}) = -a^{-1}.$

Section 4.1, Problem 13:

Show that the set of matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, wehre $a, b \in \mathbb{R}$, is a field under the operations of matrix addition and multiplication.

Hint:

We need to check all conditions in the definition of a field.

(i) Closure: check that the sum and the product of any two matrices of the above form are also of the above form.

(ii) Associative laws: we know that these hold for both operations for all matrices, so do not have to be checked again.

(iii) Commutative laws: addition of any matrices is commutative and does not have to be checked here. However, multiplication of matrices is not commutative in general, so we have to check that any two matrices of the above form commute.

(iv) Distributivity laws hold for all matrices and do not have to be checked here.

(v) Identity elements: check that both 0 and I are in the given set.

(vi) Inverse elements: check that for every nonzero matrix in the given set, both its additive and multiplicative inverses are in this set.