Math 151

Solutions to selected homework problems

Section 5.2, Problem 1:

Let R be a commutative ring, and let D be an integral domain. Let $\phi : R \to D$ be a nonzero function such that $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$, for all $a, b \in R$. Show that ϕ is a ring homomorphism.

Solution:

Since ϕ preserves both operations, but the definition of a ring homomorphism we only have to show that $\phi(1) = 1$. We have $\phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1)$, thus $\phi(1)(\phi(1)-1) = 0$. Since D is an integral domain, either $\phi(1) = 0$ or $\phi(1) = 1$.

However, we will show that $\phi(1) = 0$. Indeed, if $\phi(1) = 0$, then for any $x \in R$, $\phi(x) = \phi(x \cdot 1) = \phi(x)\phi(1) = \phi(x) \cdot 0 = 0$, so ϕ is the zero function, but it was given that ϕ is nonzero. Therefore $\phi(1) = 1$.

Section 5.2, Problem 2:

Let F be a field and let $\phi : F \to R$ be a ring homomorphism. Show that ϕ is either zero or one-to-one.

Solution:

By definition, $\phi(1) = 1$. Let's consider the following two cases:

Case I: $1 = 0$ in R. Then for any $x \in R$, $\phi(x) = \phi(x \cdot 1) = \phi(x)\phi(1) = \phi(x) \cdot 0 = 0$, so ϕ is the zero function.

Case II: $1 \neq 0$ in R. We will prove (by contradiction) that ϕ is one-to-one. Suppose not. Then $\phi(x) = \phi(y)$ for some $x \neq y$. Then $\phi(x-y) = 0$ and $x - y \neq 0$. Since F is a field, $x - y$ has a multiplicative inverse, and then $\phi(1) = \phi((x-y)(x-y)^{-1}) = \phi(x-y)\phi((x-y)^{-1}) = 0 \cdot \phi((x-y)^{-1}) = 0$. This contradicts to the assumption $\phi(1) = 1 \neq 0$. Therefore ϕ is one-to-one.

Section 5.2, Problem 5:

Show that the identity mapping is the only ring homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$.

Solution:

Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be a ring homomorphism. Then $\phi(0) = 0$ and $\phi(1) = 1$. For each positive $n \in \mathbb{Z}$, $\phi(n) = \phi(1+1+\ldots 1) = \phi(1)+\phi(1)+\ldots+\phi(1) = n\phi(1)$. For each negative $n \in \mathbb{Z}$, $-n$ is positive, and $\phi(n) = -\phi(-n) = -\phi(1+1+\ldots 1) = -(\phi(1)+\phi(1)+\ldots+\phi(1)) = -(-n) = n$. So ϕ is the identity function.

Section 5.2, Problem 11:

Show that the direct sum of two nonzero rings is never an integral domain.

Solution:

Let R and S be nonzero rings. Then there exist nonzero elements $r \in R$ and $s \in S$. Then $(r, 0)$ and $(0, s)$ are nonzero elements of $R \oplus S$, but $(r, 0)(0, s) = (0, 0)$, thus $R \oplus S$ is not an integral domain.