Math 151

Practice problems for Test 1

The actual test will consist of 6 problems. You will have 1 hour to complete the test.

1. Let $a = 51$ and $b = 85$.
   (a) Find the greatest common divisor d of a and b.
   (b) Find $m,\, n \in \mathbb{Z}$ such that $d = am + bn$.
2. Prove that if a and b are integers, then $b|a \;\Leftrightarrow\; a\mathbb{Z} \subset b\mathbb{Z}$.
3. Solve the congruences:
   (a) $15x \equiv 21 \pmod{24}$
   (b) $15x \equiv 8 \pmod{24}$
   i.e. find all distinct solutions (and explain why there are no other solutions).
4. Solve the system $x \equiv 6 \pmod{25}$, $x \equiv 2 \pmod{11}$.
5. (a) Give the definition of $\phi(n)$.
   (b) Find $\phi(15)$.
   (c) Prove that if p and q are distinct primes, then $\phi(pq) = pq - p - q + 1$.
6. Find the multiplicative order and inverse of $[101]$ in $\mathbb{Z}_{1000}$.
7. Which of the following are well-defined functions? For each well-defined function, determine whether it is one-to-one and/or onto.
   (a) $f : \mathbb{Z}_8 \to \mathbb{Z}_{12}$ given by $f([x]_8) = [3x]_{12}$
   (b) $f : \mathbb{Z}_8 \to \mathbb{Z}_{16}$ given by $f([x]_8) = [-x]_{16}$
8. Define $f : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $f([x]_{mn}) = ([x]_m, [x]_n)$. Show that f is a function and that f is onto if and only if $\gcd(m, n) = 1$.
9. Consider the set of real numbers $\mathbb{R}$. Which of the following are equivalence relations? For those which are, show that all the conditions for an equivalence relation (reflexive, symmetric, and transitive laws) are satisfied, and describe the equivalence classes. For those which are not, say which condition(s) is (are) not satisfied, and why (give a specific example to show that a condition is not satisfied).
   (a) $x \sim y$ if $|x| = |y|$
   (b) $x \sim y$ if $xy > 0$
   (c) $x \sim y$ if either $xy > 0$ or $x = y = 0$.
10. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 4 & 5 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 2 & 3 & 6 \end{pmatrix}$.
    (a) Compute $\sigma\tau$ and $\tau\sigma$.
    (b) Do σ and τ commute?
    (c) Find σ^{-1} and τ^{-1}.
    (d) Write σ and τ as products of disjoint cycles.
    (e) Draw the associated diagrams for σ and τ.
    (f) Write σ as a product of transpositions.
    (g) Are σ and τ even or odd permutations?