Math 151 Fall 2008

Practice problems for Test 3

The actual test will consist of 6 problems. You will have 1 hour to complete the test.

1. Prove that the intersection of two normal subgroups is normal.
2. Let $G = GL_2(\mathbb{R})$, let H be the set of upper triangular invertible matrices, and let K be the set of upper triangular invertible matrices with 1's on the diagonal. Then $K \subseteq H \subseteq G$. Is H normal in G? Is K normal in H? Is K normal in G?
3. Let $\mathbb{Z}_2(i) = \{a + bi \mid a, b \in \mathbb{Z}_2\}$ with addition and multiplication given by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ab - bd) + (ad + bc)i$$

   where $a, b, c, d \in \mathbb{Z}_2$. Is $\mathbb{Z}_2(i)$ a ring? If so, is it a commutative ring? Is it an integral domain? Is it a field? (Justify your answers: say which axioms are or are not satisfied, and why.)
4. Find the quotient and remainder when $x^5 + 3x + 1$ is divided by $x^2 + 2$ over $\mathbb{Q}$.
5. Let $f(x) = x^5 + 4x^4 + 6x^3 + 6x^2 + 5x + 2$ and $g(x) = x^4 + 3x^2 + 3x + 6$.
   (a) Find the greatest common divisor $d(x)$ of $f(x)$ and $g(x)$ over $\mathbb{Z}_7$.
   (b) Find polynomials $a(x)$ and $b(x)$ in $\mathbb{Z}_7[x]$ such that $d(x) = a(x)f(x)+b(x)g(x)$.
6. Find the multiplicative inverse of $[x + 4]$ in $\mathbb{Z}_5/< x^3 + x + 1 >$.
7. Find all integer roots of $x^4 + 4x^3 + 8x + 32 = 0$.
8. Find the irreducible factors of $x^3 - 2$ over $\mathbb{Z}$; $\mathbb{Q}$; $\mathbb{R}$; $\mathbb{C}$; $\mathbb{Z}_3$.
9. Find all irreducible polynomials of degree 3 over $\mathbb{Z}_2$.
10. Use Eisenstein's criterion to show that the polynomial $3x^4+30x-60$ is irreducible over $\mathbb{Q}$.
11. Find all units in
    (a) $\mathbb{Z}_6 \oplus \mathbb{Z}_8$
    (b) $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
    (c) $\mathbb{R} \oplus \mathbb{R}$