Math 151

Project B (optional, for extra credit)

Last day to submit solutions: 8 October 2008.

Note: Giving answers "yes" or "no" and giving examples without explanations is not sufficient. Provide proofs for all your statements.

Definition. An $n \times 1$ matrix $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is called a vector of length n. The set of all vectors of length n with real entries is denoted by $\mathbb{R}^n$. The set of all vectors of length n with integer coefficients is denoted by $\mathbb{Z}^n$. If M is an $m \times n$ matrix and v is a vector of length n, then the product Mv is a vector of length m. The function $T_M : \mathbb{R}^n \to \mathbb{R}^m$ given by $T_M(v) = Mv$ is called a linear transformation defined by the matrix M.

1. Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $T_M : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by M, that is, T_M is the multiplication by M on the left. Is T_M one-to-one? Is T_M onto?
2. For which (find a necessary and sufficient condition) matrices M in $\mathrm{Mat}_{n\times n}(\mathbb{R})$ is $T_M : \mathbb{R}^n \to \mathbb{R}^n$ one-to-one and onto?
3. Let $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ again. Notice that if $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ is such that x and y are integers, that is, if $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{Z}^2$, then $M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3y + 4y \end{bmatrix} \in \mathbb{Z}^2$. Thus we can restrict T_M onto $\mathbb{Z}^2$, and consider the map $T_M : \mathbb{Z}^2 \to \mathbb{Z}^2$.
   (a) Is $T_M : \mathbb{Z}^2 \to \mathbb{Z}^2$ one-to one?
   (b) Is $T_M : \mathbb{Z}^2 \to \mathbb{Z}^2$ onto?
   (c) If the answer in part (a) is "yes", give an example of a $2 \times 2-$matrix M' s.t. $T_{M'} : \mathbb{Z}^2 \to \mathbb{Z}^2$ is not one-to-one. If the answer in part (a) is "no", give an example of M' s.t. $T_{M'} : \mathbb{Z}^2 \to \mathbb{Z}^2$ is one-to-one.
   (d) If the answer in part (b) is "yes", give an example of a $2 \times 2-$matrix M'' s.t. $T_{M''} : \mathbb{Z}^2 \to \mathbb{Z}^2$ is not onto. If the answer in part (b) is "no", give an example of M'' s.t. $T_{M''} : \mathbb{Z}^2 \to \mathbb{Z}^2$ is onto.
4. Let $a,\ b,\ c$, and d be integers, and let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
   (a) Find a necessary and sufficient condition on $a,\ b,\ c$, and d for $T_M : \mathbb{Z}^2 \to \mathbb{Z}^2$ to be one-to-one.
   (b) Find a necessary and sufficient condition on $a,\ b,\ c$, and d for $T_M : \mathbb{Z}^2 \to \mathbb{Z}^2$ to be onto.
5. Generalize your results in problem 4 to the case of $n\times n$ matrices with integer coefficients. Warning: this problem requires more linear algebra than reviewed in this class. It is probably too hard if you haven't taken Linear Algebra.