Practice problems for Test 2

Solutions

1. (Note: feel free to show me your examples to make sure they are correct.)

<table>
<thead>
<tr>
<th>group</th>
<th>order</th>
<th>abelian?</th>
<th>cyclic?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_4 )</td>
<td>4</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( \mathbb{Z}_6 )</td>
<td>6</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( S_3 )</td>
<td>6</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>( \mathbb{Z}_4 \oplus \mathbb{Z}_2 )</td>
<td>8</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>( \mathbb{Z} )</td>
<td>( \infty )</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( GL_2(\mathbb{R}) )</td>
<td>( \infty )</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>{e} = trivial</td>
<td>1</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( D_5 )</td>
<td>10</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>( Mat_{2 \times 3}(\mathbb{Z}_2) )</td>
<td>64</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>( \text{Vec} )</td>
<td>( \infty )</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

\( \mathbb{Z}_4^* \) consists of all units in \( \mathbb{Z}_4 \), and it is a group under multiplication. \( \mathbb{Z}_4^* = \{1, 2, 3, 4\} \), so \( |\mathbb{Z}_4^*| = 4 \). It is abelian since multiplication of numbers is commutative. It is cyclic because it is generated by \( 2 < 2 > = \{1, 2, 4, 3\} = \mathbb{Z}_4^* \).

\( \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\} = \langle 1 \rangle > \) is an abelian cyclic group (under addition) of order 6. (In general, \( \mathbb{Z}_n = \{0, 1, 2, \ldots , n - 1\} = \langle 1 \rangle > \) is an abelian cyclic group of order \( n \).)

\( S_6 = \{(1), (12), (13), (23), (123), (132)\} \) has order 6. It is not abelian because e.g. \( (12)(13) = (123) \) and \( (13)(12) = (123) \). (In general, \( S_n \), the permutation group on a set of \( n \) elements, has order \( n! \) and is non-abelian.) It is not cyclic because every cyclic group is abelian and this one is not.

\( \mathbb{Z}_4 \oplus \mathbb{Z}_2 = \{ (x, y) \mid x \in \mathbb{Z}_4, y \in \mathbb{Z}_2 \} \) is the set of all pairs, and it has order \( 4 \cdot 2 = 8 \). It is abelian since both \( \mathbb{Z}_4 \) and \( \mathbb{Z}_2 \) are. It is not cyclic because it has no element of order 8; it is easy to check that the order of each element is \( \leq 4 \).

\( \mathbb{Z} \) is an infinite cyclic group consisting of all integer numbers (with addition). It is abelian since addition of numbers is abelian. It is cyclic because it is generated by 1.

\( GL_2(\mathbb{R}) \) is the group of \( 2 \times 2 \) invertible matrices with real entries under multiplication. There are infinitely many such matrices, so its order is infinity. It is not abelian because e.g. \[
\begin{bmatrix}
0 & 2 \\
3 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
= \begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}
\begin{bmatrix}
0 & 2 \\
1 & 0
\end{bmatrix}
= \begin{bmatrix}
3 & 0 \\
0 & 2
\end{bmatrix}.
\]
It is not cyclic because every cyclic group is abelian.

\( \{e\} \) = trivial group has only one element. It is abelian (all elements commute), and cyclic (generated by \( e \)). It is a very uninteresting group, but I just wanted to give an example of a group of order 1.

\( D_5 \), dihedral group of order \( 2 \cdot 5 = 10 \), is the group of rigid motions of a regular pentagon. Its elements are \( e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b \). It is not abelian (e.g. \( ba = a^4b \neq ab \)), and hence not cyclic.

\( Mat_{2 \times 3}(\mathbb{Z}_2) \) is the group of all \( 2 \times 3 \) matrices with entries in \( \mathbb{Z}_2 \) under addition. Its order is 64: each entry can be either 0 or 1, and there are 6 entries, so there are \( 2^6 = 64 \) such matrices. It is abelian since addition in \( \mathbb{Z}_2 \) is commutative. But it is not cyclic: each non-zero element has order 2 because if you add an entry of a matrix to itself you’ll get 0, thus any matrix added to itself gives the zero matrix. Therefore there is no element (matrix) of order 64.
\( \mathbb{R} \) is an infinite group of real numbers under addition. It is abelian (addition of numbers is commutative) but not cyclic: every nonzero element generates a cyclic subgroup consisting of its own multiples, thus every cyclic subgroup has a smallest positive element. But \( \mathbb{R} \) does not have any.

2. First notice that \( \mathbb{R}, \mathbb{R}^* \), and \( \mathbb{R}^+ \) have infinite order, while \( \mathbb{Z}_4 \oplus \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z}_8 \oplus \mathbb{Z}_2, \) and \( \mathbb{Z}_{16} \) have order 16. So we only have to check the first 3 groups, and the last 4 groups, separately.

Among the first 3 groups, \( \mathbb{R} \) and \( \mathbb{R}^+ \) are isomorphic: let \( f : \mathbb{R} \to \mathbb{R}^+ \) be defined by \( f(x) = e^x \). Then \( f(x + y) = e^{x+y} = e^x e^y = f(x)f(y) \). \( f \) is one-to-one because if \( f(x) = f(y) \) then \( e^x = e^y \), then \( \ln e^x = \ln e^y \) which implies \( x = y \). Finally, \( f \) is onto because for any positive real \( z \), let \( x = \ln z \), then \( f(x) = f(\ln z) = e^{\ln z} = z \). (See example 3.4.2 on p.115.)

\( \mathbb{R} \) and \( \mathbb{R}^* \) are not isomorphic because the first group has no element of order 2, and the second group has an element of order 2, namely \(-1 : (-1)^2 = 1\). \( \mathbb{R}^+ \) and \( \mathbb{R}^* \) are not isomorphic for the same reason. (See example 3.4.3 on p.116.)

Among the last 4 groups, \( \mathbb{Z}_2 \oplus \mathbb{Z}_8 \) and \( \mathbb{Z}_8 \oplus \mathbb{Z}_2 \) are isomorphic: define \( f : \mathbb{Z}_2 \oplus \mathbb{Z}_8 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \) by \( f((x,y)) = (y,x) \). Obviously this is a 1-1 correspondence, and it is a homomorphism because \( f((x,y)+(z,w)) = f((x+z,y+w)) = (y+w,x+z) = (y,x)+(w,z) = f((x,y))+f((z,w)) \).

All other pairs are not isomorphic: \( \mathbb{Z}_2 \oplus \mathbb{Z}_4 \) only has elements of order \( \leq 4 \); \( \mathbb{Z}_2 \oplus \mathbb{Z}_8 \) and \( \mathbb{Z}_8 \oplus \mathbb{Z}_2 \) have elements of order 8 but no elements of order 16; \( \mathbb{Z}_{16} \) has elements of order 16.

3. Let’s denote this subset of \( G \) by \( H \). We want to show that \( H \) is a subgroup.

Closed under multiplication: if \( a, b \in H \), then \( a^2 = b^2 = e \). Then \( (ab)^2 = a^2 b^2 = e \cdot e = e \), so \( ab \in H \).

Identity: \( \text{ord}(e) = 1 \leq 2 \), so \( e \in H \).

Closed under inverses: if \( a \in H \), then \( a^2 = e \). Then \( (a^{-1})^2 = (a^2)^{-1} = e^{-1} = e \), so \( a^{-1} \in H \).

4. Let’s denote the given matrix by \( A \). We have to compute powers of \( A \) until we get the identity matrix. The smallest positive \( k \) such that \( A^k = I \) is then the order of \( A \), and the cyclic subgroup generated by \( A \) is \( \{I, A, A^2, \ldots, A^{k-1}\} \). Notice that entries of our matrices are elements of \( \mathbb{Z}_3 \), so each time we multiply matrices, we have to reduce each entry of the product modulo 3. Then

\[
< \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} > = \left\{ I_2, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \right\}.
\]

Therefore the order of \( A \) is 6.

5. (a) Generators of \( \mathbb{Z}_{24} \) are numbers (more precisely, classes of numbers) between 0 and 24 that are relatively prime to 24. There are 8 of them: 1, 5, 7, 11, 13, 17, 19, 23.

(b) \( H = \{0,6,12,18\} \) is a cyclic subgroup. Generators: 6 and 18. 0 and 12 are not generators because the order of 0 is 1, and the order of 12 is 2.

\( K = \{0,4,8,12,16,20\} \) is a cyclic subgroup. Generators: 4 and 20.

\( H \cap K = \{0,12\} \) is a cyclic subgroup. Generator: 12.

\( H \cup K = \{0,4,6,8,12,16,18,20\} \) is not a subgroup: it is not closed under addition, e.g.,

\( 4 \neq 6 \in H \cup K \).

\( H + K = \{0,2,4,6,8,10,12,14,16,18,20\} \) is a cyclic subgroup. Generators: 2, 10, 14, 22.

6. (a) \( f : \mathbb{Z} \to \mathbb{Z}, \ f(x) = 3x \) is a homomorphism: \( f(x+y) = 3(x+y) = 3x + 3y = f(x) + f(y) \).

\( \text{Ker}(f) = \{0\} \). Image \( = 3\mathbb{Z} \), the set of all multiples of 3. It is one-to-one because \( 3x = 3y \) implies \( x = y \). It is not onto because e.g. 1 is not in the image.
(b) \( f : Z \to Z_4 \) is a homomorphism: \( f(x + y) = [x + y]_4 = [x]_4 + [y]_4 = f(x) + f(y) \).
\( \ker(f) = 4Z \), the set of all multiples of 4. \( \text{Image} = Z_4 \). It is not one-to-one because e.g. \( f(0) = [0]_4 \) and \( f(4) = [4]_4 = [0]_4 \). It is onto: every element of \( Z_4 \) is in the image since \([x]_4 = f(x)\).

(c) \( f : Z \to Z_6 \) is a homomorphism: \( f(x + y) = [2(x + y)]_6 = [2x + 2y]_6 = [2x]_6 + [2y]_6 = f(x) + f(y) \).
\( \ker(f) = 3Z \), the set of all multiples of 3. \( \text{Image} = Z_6 \). It is not one-to-one because e.g. \( f(0) = [0]_6 \) and \( f(3) = [6]_6 = [0]_6 \). It is not onto because e.g. \([1]_6 \) is not in the image.

(d) \( f : Z_2 \to Z \), \( f([x]_2) = x \) is not a homomorphism because it is not a well-defined function: \( [0]_2 = [2]_2 \), but \( f([0]_2) = 0 \), \( f([2]_2) = 2 \), and \( 0 \neq 2 \).

(e) \( f : \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \), \( f((x, y)) = x + y \) is a homomorphism. The kernel consists of all pairs \((x, y)\) for which \( x + y = 0 \). \( \ker(f) = \{(x, -x)\} \). \( \text{Image} = \mathbb{R} \). \( z = f((z, 0)) \) is in the image. It is not one-to-one because e.g. \( f((1, 0)) = 1 \) and \( f((2, 1)) = 1 \).

(f) \( f \) is one-to-one because the kernel is trivial, and it is not onto because e.g. \( \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \) is not in the image.

7. Let \( H \) and \( K \) be normal in \( G \). We want to show that \( H \cap K \) is normal (we already know that it is a subgroup: there was a homework problem in which we proved that the intersection of any collection of subgroups is a subgroup). Let \( x \in H \cap K \), and let \( g \in G \). Then \( x \in H \) and \( x \in K \). Since \( H \) is normal, \( gxg^{-1} \in H \). Since \( K \) is normal, \( gxg^{-1} \in K \). Then \( gxg^{-1} \in H \cap K \), therefore \( H \cap K \) is normal.

8. \( H \) is not normal in \( G \) because e.g. \( \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \notin H \).

\( K \) is normal in \( H \) because for any \( \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in K \) and \( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in H \),
\( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b + ax + b \\ 0 & c \end{bmatrix} \) is in \( K \).

\( K \) is not normal in \( G \) because the counterexample above works in this case too.