Practice problems for Test 2

The actual test will consist of 6 problems. You will have 1 hour to complete the test.

1. Fill in the table below and provide brief explanations. For the last 4 rows, give your own examples of groups, determine their order, whether they are abelian, and whether they are cyclic.

group	order	abelian?	cyclic?
$\mathbb{Z}_5^*$			
	6	yes	
	6	no	
	8	yes	no
	∞		yes
	∞	no	

2. Which of the following groups are isomorphic? $\mathbb{R}$, $\mathbb{R}^*$, $\mathbb{R}^+$, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_8$, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$, $\mathbb{Z}_{16}$
3. Let G be an abelian group. Prove that the set of elements of G of order less than or equal to 2 is a subgroup.
4. Find the order and the cyclic subgroup generated by $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ in $GL_2(\mathbb{Z}_3)$.
5. Let $G = \mathbb{Z}_{24}$.
   (a) How many generators does G have?
   (b) Consider subgroups $H =< 6 >$ and $K =< 4 >$. List all the elements of H and K. Find $H \cap K$, $H \cup K$, and $H + K$. Which of these are subgroups of G? For those which are subgroups, are they cyclic? If so, find all the generators.
6. Which of the following are homomorphisms? (Explain why or why not.) For each homomorphism, determine its kernel, image, and whether it is one-to-one or onto.
   (a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x$
   (b) $f : \mathbb{Z} \to \mathbb{Z}_4$, $f(x) = [x]_4$
   (c) $f : \mathbb{Z} \to \mathbb{Z}_6$, $f(x) = [2x]_6$
   (d) $f : \mathbb{Z}_2 \to \mathbb{Z}$, $f([x]_2) = x$
   (e) $f : \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$, $f((x, y)) = x + y$
   (f) $f : \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$, $f((x, y)) = xy$
   (g) $f : \mathbb{R}^* \times \mathbb{R}^* \to GL_2(\mathbb{R})$, $f((x, y)) = \begin{bmatrix} 2x - y & y - x \\ 2x - 2y & 2y - x \end{bmatrix}$
7. Prove that the intersection of two normal subgroups is normal.
8. Let $G = GL_2(\mathbb{R})$, let H be the set of upper triangular invertible matrices, and let K be the set of upper triangular invertible matrices with 1's on the diagonal. Then $K \subset H \subset G$. Is H normal in G? Is K normal in H? Is K normal in G?