Math 151 Spring 2004

Test 1 - Solutions

1. Let a, b, $c \in \mathbb{Z}$, $c \neq 0$. Prove that $bc|ac \;\Leftrightarrow\; b|a$.
   ($\Rightarrow$) *If $bc|ac$ then $ac = mbc$ for some integer m. Since $c \neq 0$, $a = mb$, i.e. $b|a$.*
   ($\Leftarrow$) *If $b|a$ then $a = mb$ for some integer m. Then $ac = mbc$, i.e. $bc|ac$.*

2. Solve the congruence $30x \equiv 18 (\text{mod } 27)$.
   Since $(30, 27) = 3|18$, the congruence has 3 distinct solutions modulo 27, which are congruent modulo 9.
   Divide by 3: $10x \equiv 6(\text{mod } 9)$. Now there are at least 2 different approaches.
   Approach 1: Since $10 \equiv 1(\text{mod } 9)$, the equation in equivalent to $x \equiv 6(\text{mod } 9)$.
   Approach 2 (the more standard one): Now $(10, 9) = 1$, so the congruence has a unique solution modulo 9. To find a solution, we will find integers a and b such that $10a = 6+9b$, or $6 = 10a+9(-b)$. First, $1 = 10+9(-1)$ can be found using the Euclidean algorithm or simply by observation since the numbers are small. Now multiply both sides by 6: $6 = 10 \cdot 6 + 9(-6)$.
   Then $10 \cdot 6 \equiv 6(\text{mod } 9)$, so $x = 6$ is a solution, so the answer is $x \equiv 6(\text{mod } 9)$.

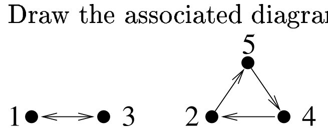
3. Find
   (a) the multiplicative order
   (b) the multiplicative inverse

   of $[3]$ in $\mathbb{Z}_{11}^*$.
   $3^2 = 9$
   $3^3 = 27 \equiv 5(\text{mod } 11)$
   $3^4 \equiv 5 \cdot 3 = 15 \equiv 4(\text{mod } 11)$
   $3^5 \equiv 4 \cdot 3 = 12 \equiv 1(\text{mod } 11)$
   Therefore the multiplicative order of $[3]_{11}$ is 5, and the multiplicative inverse of $[3]_{11}$ is $[4]_{11}$.

4. Is $f : \mathbb{Z}_{12} \to \mathbb{Z}_8$ given by $f([x]_{12}) = [3x]_8$ a well-defined function? Explain why or why not.
   No, because e.g. $[0]_{12} = [12]_{12}$ but $f([0]_{12}) = [0]_8$ and $f([12]_{12}) = [3 \cdot 12]_8 = [36]_8 = [4]_8$.

5. Consider the set of real numbers $\mathbb{R}$. For x and y in $\mathbb{R}$, let $x \sim y$ if $(x - y) \in \mathbb{Z}$. Show that $\sim$ is an equivalence relation, and describe the equivalence classes.
   Reflexive law: for each x, $x \sim x$ since $x - x = 0 \in \mathbb{Z}$.
   Symmetric law: if $x \sim y$, then $(x - y) \in \mathbb{Z}$, then $(y - x) = -(x - y) \in \mathbb{Z}$, so $y \sim x$.
   Transitive law: if $x \sim y$ and $y \sim z$, then $(x-y) \in \mathbb{Z}$ and $(y-z) \in \mathbb{Z}$, then $x-z = (x-y)+(y-z) \in \mathbb{Z}$, so $x \sim z$.
   The equivalence class of x is the set of all real numbers y such that $y - x = m \in Z$, i.e. $y = x + m$: $[x] = \{\ldots, x-3, x-2, x-1, x, x+1, x+2, x+3, \ldots\}$. There are infinitely many equivalence classes, one class for each number $a \in [0, 1)$.

6. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix}$.
   (a) Find $\tau\sigma$.
   $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$
   (b) Draw the associated diagram for σ.

   (c) Write σ as a product of disjoint cycles.
   $\sigma = (13)(254)$

Optional: Does there exist an integer number m such that for any prime number p, $m \equiv p - 1(\text{mod } p)$? If such a number exists, find it. If not, prove that there is no such number.

Yes. $m = -1$ satisfies that property.