Math 151 Spring 2004

Test 2 - Solutions

1. Fill in the table. Provide brief explanations.

group	order	abelian?	cyclic?
$\mathbb{Z}_9^*$	*6*	*yes*	*yes*
$\mathbb{R}$	∞	yes	no
S_4	24	no	*no*

$\mathbb{Z}_9^* = \{1, 2, 4, 5, 7, 8\}$, so $|\mathbb{Z}_9^*| = 6$. It is abelian since multiplication of numbers is commutative. It is cyclic because $< 2 >= \mathbb{Z}_9^*$.
$|\mathbb{R}| = \infty$, it is abelian since addition of numbers is commutative. It is not cyclic because it has no smallest positive element, and every cyclic subgroup $\{\ldots, -2x, -x, 0, x, 2x, \ldots\}$ has a smallest positive element. Other examples of abelian non-cyclic groups: $\mathbb{R}^*$, $\mathbb{R}^+$, $\mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z}_n \oplus \mathbb{Z}_m$ for any $n, m \geq 2$, $\mathbb{Z}_{12}^*$ (and also some other $\mathbb{Z}_n^*$), $Mat_{m \times n}(\mathbb{R})$.
$|S_4| = 4! = 24$. It is not abelian because e.g. $(12)(23) = (123)$ and $(23)(12) = (132)$. It is not cyclic since every cyclic group is abelian. Other examples of non-abelian groups of order 24: D_{12}, $S_3 \times \mathbb{Z}_4$, $S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $D_4 \times \mathbb{Z}_3$.

2. Are any of the groups $\mathbb{Z}_2 \oplus \mathbb{Z}_5$, $\mathbb{Z}_{10}$, $\mathbb{Z}_{10}^*$, D_5 isomorphic? Explain.
First find some properties of the given groups:

group	*order*	*abelian?*	*cyclic?*
$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	*10*	*yes*	*yes*
$\mathbb{Z}_{10}$	*10*	*yes*	*yes*
$\mathbb{Z}_{10}^*$	*4*	*yes*	*yes*
D_5	*10*	*no*	*no*

We see that $\mathbb{Z}_{10}^*$ is the only group of order 4, and D_5 is the only nonabelian group (among the given ones). Thus neither of them is isomorphic to any other of the given groups.
In $\mathbb{Z}_2 \oplus \mathbb{Z}_5$, $ord((1,1)) = 10$, thus $\mathbb{Z}_2 \oplus \mathbb{Z}_5$ is cyclic of order 10. Therefore it is isomorphic to $\mathbb{Z}_{10}$.

3. Find the order of $(2, 5)$ in $\mathbb{Z}_3 \oplus \mathbb{Z}_{10}$.
The identity element of $\mathbb{Z}_3 \oplus \mathbb{Z}_{10}$ is $(0, 0)$.
$(2, 5) \neq (0, 0)$,
$2 \cdot (2, 5) = (4, 10) = (1, 0) \neq (0, 0)$,
$3 \cdot (2, 5) = (6, 15) = (0, 5) \neq (0, 0)$,
$4 \cdot (2, 5) = (8, 20) = (2, 0) \neq (0, 0)$,
$5 \cdot (2, 5) = (10, 25) = (1, 5) \neq (0, 0)$,
$6 \cdot (2, 5) = (12, 30) = (0, 0) = (0, 0)$, *so* $ord((2, 5)) = 6$.

4. Show that the function $f : \mathbb{R} \to \mathbb{R}^*$ defined by $f(x) = 2^x$ is a homomorphism.
$f(x + y) = 2^{x+y} = 2^x \cdot 2^y = f(x) \cdot f(y)$, so f preserves the operation (addition in $\mathbb{R}$, multiplication in $\mathbb{R}^*$).

5. Find the kernel and the image of the homomorphism
$f : \mathbb{Z}_{10} \to \mathbb{Z}_8$ defined by $f([x]_{10}) = [4x]_8$.
$f([0]_{10}) = [0]_8$,
$f([1]_{10}) = [4]_8$,
$f([2]_{10}) = [8]_8 = [0]_8$,
$f([3]_{10}) = [12]_8 = [4]_8$,
$f([4]_{10}) = [16]_8 = [0]_8$,
$f([5]_{10}) = [20]_8 = [4]_8$,
$f([6]_{10}) = [24]_8 = [0]_8$,
$f([7]_{10}) = [28]_8 = [4]_8$,
$f([8]_{10}) = [32]_8 = [0]_8$,
$f([9]_{10}) = [36]_8 = [4]_8$.
$Ker(f) = \{[x]_{10} \mid f([x]_{10}) = [0]_8\} = \{[0]_{10}, [2]_{10}, [4]_{10}, [6]_{10}, [8]_{10}\} = 2\mathbb{Z}_{10}$.
$Image(f) = \{[y]_8 \mid [y]_8 = f([x]_{10})\ for\ some\ x\} = \{[0]_8, [4]_8\} = 4\mathbb{Z}_8$.

6. Let $G = GL_2(\mathbb{R})$ and $H = \{M \in G \mid \det(M) > 0\}$.

- Show that H is a subgroup of G.
  Closed under multiplication: if $M_1, M_2 \in H$, i.e. $det(M_1) > 0$, $det(M_2) > 0$, then $det(M_1M_2) = det(M_1)det(M_2) > 0$, so $M_1M_2 \in H$.
  Identity: $det(I_2) = 1 > 0$, so $I_2 \in H$.
  Closed under inverses: if $M \in H$, i.e. $det(M) > 0$, then $det(M^{-1}) = \frac{1}{\det(M)} > 0$, so $M^{-1} \in H$.

- Show that H is normal in G.
  Let $M \in H$ (so $det(M) > 0$), and $N \in G$. Then $det(NMN^{-1}) = det(N)det(M)det(N^{-1}) = det(N)det(M)\frac{1}{\det(N)} = det(M) > 0$, so $NMN^{-1} \in H$. Therefore H is normal.

Optional: Are $D_3 \times \mathbb{Z}_4$ and $D_4 \times \mathbb{Z}_3$ isomorphic?

Both groups have order 24, are nonabelian, noncyclic, and have elements of orders 1, 2, 3, 4, 6, and 12. But let's count the number of elements of each order in each group.

D_3 has 6 elements, their orders are 1, 2, 2, 2, 3, 3.

$\mathbb{Z}_4$ has 4 elements, their orders are 1, 2, 4, 4.

D_4 has 8 elements, their orders are 1, 2, 2, 2, 2, 2, 4, 4.

$\mathbb{Z}_3$ has 3 elements, their orders are 1, 3, 3.

Using the fact that if $x \in G$ and $y \in H$, then the order of (x, y) in $G \times H$ is the least common multiple of the order of x in G and the order of y in H, we find the number of elements of orders 1, 2, 3, 4, 6, and 12 in each group of the given groups:

group	*order 1*	*order 2*	*order 3*	*order 4*	*order 6*	*order 12*
$D_3 \times \mathbb{Z}_4$	*1*	*7*	*2*	*8*	*2*	*4*
$D_4 \times \mathbb{Z}_3$	*1*	*5*	*2*	*2*	*10*	*4*

We see that these groups have different number of elements of order 2, 4, and 6, therefore they are not isomorphic.