Math 151

Brief Review of Matrices

Definition 1. An $m \times n$ matrix is an array of numbers (or polynomials, or any functions, or elements of any algebraic structure...) with m rows and n columns. In this handout, all entries of a matrix are assumed to be real numbers. Later in this course we will discuss matrices with integer, rational, and other entries. The entry in the i-th row and j-th column of a matrix A is denoted by a_{ij}.

Example 2. $A = \begin{bmatrix} -5 & 0 & 2 \\ 4 & 3 & 0 \end{bmatrix}$ is a 2×3 matrix with $a_{11} = -5$, $a_{12} = 0$, $a_{13} = 2$, $a_{21} = 4$, $a_{22} = 3$, and $a_{23} = 0$.

Definition 3. If $m = n$, i.e. the number of rows is equal to the number of columns, the matrix is said to be a square matrix. The elements $a_{11}, a_{22}, \ldots, a_{nn}$ are called the diagonal elements.
A matrix A is called diagonal if $a_{ij} = 0$ for all $i \neq j$ (so all the nondiagonal entries are zero).
A matrix A is called symmetric if $a_{ij} = a_{ji}$ for all i and j. A is called antisymmetric if $a_{ij} = -a_{ji}$ (in particular, all diagonal entries are 0).
A matrix A is called upper triangular if $a_{ij} = 0$ for all $i > j$. A is lower triangular if $a_{ij} = 0$ for all $i < j$.

Example 4.

$$\text{diagonal: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{symmetric: } \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} \qquad \text{antisymmetric: } \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

$$\text{upper triangular: } \begin{bmatrix} 0 & 1 & -2 \\ 0 & 3 & 4 \\ 0 & 0 & -5 \end{bmatrix} \qquad \text{lower triangular: } \begin{bmatrix} 6 & 0 \\ -7 & 8 \end{bmatrix}$$

Definition 5. A matrix is said to be in Row Echelon Form (REF) if the first nonzero element of each row is to the right of the first nonzero element of the previous row; all the zero rows are at the bottom of the matrix. A matrix is said to be in Reduced Row Echelon Form (RREF) if it is in REF, and the first nonzero element of each row is 1 which is the only nonzero element in its column.

Example 6.

$$\text{REF but not RREF: } \begin{bmatrix} 4 & 1 & 1 & 2 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{RREF: } \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row operations. Let c be a nonzero real number.

1. $A_{i,j}(c)$: add c times i-th row to the j-th row.

2. $S_{i,j}$: switch the i-th and j-th rows.

3. $M_i(c)$: multiply the i-th row by c.

Theorem 7. *Every matrix can be transformed into REF by the first two row operations. Every matrix can be transformed into RREF by all three row operations.*

Example 8. $\begin{bmatrix} 0 & 0 & 2 & -10 \\ 1 & 2 & 1 & 2 \\ -3 & -6 & 1 & -26 \end{bmatrix} \xrightarrow{S_{1,2}} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 2 & -10 \\ -3 & -6 & 1 & -26 \end{bmatrix} \xrightarrow{A_{1,3}(3)} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 2 & -10 \\ 0 & 0 & 4 & -20 \end{bmatrix}$

$\xrightarrow{A_{2,3}(-2)} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 2 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{A_{2,1}(-\frac{1}{2})} \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{M_2(\frac{1}{2})} \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Matrix operations. Let A, B, and C be matrices, and let c be a number.

- If A and B have the same size, $A+B$ is the matrix whose each entry is the sum of the corresponding entries in A and B. Similarly, $A-B$ is the matrix whose each entry is the difference of the corresponding entries in A and B.

- cA is the matrix whose each entry is c times the corresponding entry in A. $(-1)A$ is denoted by $-A$.

- The zero matrix is a matrix with all entries 0. The zero matrix is denoted by 0 (or $0_{m,n}$).

- Matrix addition, subtraction, and multiplication by a constant, have all the properties of the corresponding operations of numbers, such as associativity $(A+B)+C = A+(B+C)$, commutativity $A+B = B+A$, additive identity $A+0 = 0+A = A$, distributivity $c(A \pm B) = cA \pm cB$, etc. Also, since $A+(-A) = (-A)+A = 0$, the matrix $-A$ is called the additive inverse of A.

- If A is $m \times n$ and B is $n \times p$, then AB is the $m \times p$ matrix whose ij-th entry is $\sum_{k=1}^{n} a_{ik}b_{kj}$.

- Matrix multiplication is associative ($(AB)C = A(BC)$) but not commutative ($AB \neq BA$ in general). Distributivity laws hold ($A(B+C) = AB+AC$, etc.).

- The $n \times n$ matrix with 1's on the diagonal and all other entries 0 is called the identity $n \times n$ matrix and is denoted by I (or I_n).

- If A is any $m \times n$ matrix, then $AI_n = I_mA = A$.

- Let A be a square matrix. If there exists a matrix B such that $AB = BA = I$ then B is called the multiplicative inverse of A, and is denoted by A^{-1}. In this case, A is called invertible. Not every matrix is invertible.

- You cannot divide! I.e. $AB = AC$ does not imply that $B = C$. However, if A is invertible, then there exists A^{-1} such that $A^{-1}A = I$. Then $AB = AC$ implies $A^{-1}AB = A^{-1}AC$, or $IB = IC$, or $B = C$.

Example 9. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -2 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 0 & 1 \end{bmatrix}$.

Then $A + B = \begin{bmatrix} 1 & 1 & 4 \\ 6 & 3 & 4 \end{bmatrix}$, $3A = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 0 \end{bmatrix}$, $-A = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & 0 \end{bmatrix}$,

$AC = \begin{bmatrix} 1\cdot 2 + 2\cdot 4 + 3\cdot 0 & 1\cdot 3 + 2\cdot 5 + 3\cdot 1 \\ 4\cdot 2 + 5\cdot 4 + 0\cdot 0 & 4\cdot 3 + 5\cdot 5 + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} 10 & 16 \\ 28 & 37 \end{bmatrix}$.

Definition 10. If A is a square matrix, let A_R be its REF. Then the product of the diagonal entries of A_R times -1 raised to the number of row switches needed to get A_R from A is called the determinant of A. (Note: there are many different ways to define the determinant of a matrix, and many ways to compute it. We'll adopt the above definition because it provides a convenient (for our course) way to compute the determinant. However, we will mostly be working with 2×2 and 3×3 matrices. It can be shown that the determinant of 2×2 and 3×3 matrices can be computed using the following formulas:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - afh - bdi$$

Also, the determinant of an upper triangular or a lower triangular matrix is equal to the product of its diagonal entries.

Example 11. $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$.

$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & 6 & -7 \\ 0 & 0 & 8 & -9 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} = 1 \cdot (-5) \cdot 8 \cdot \frac{1}{2} = -20$.

Theorem 12. $\det(AB) = \det(A) \cdot \det(B)$.

Note. $\det(A + B) \neq \det(A) + \det(B)$ in general.

Theorem 13. *A square matrix is invertible if and only if its determinant is not equal to 0.*

Theorem 14. *If the rows (or columns) of a matrix are linearly dependent, that is, if one row is a linear combination of other rows (or one column is a linear combination of other columns), then the determinant of the matrix is 0, and thus the matrix is not invertible. In particular, if a matrix has a zero row or column, then its determinant is 0, and thus the matrix is not invertible.*

Example 15. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is not invertible because its determinant is 0, or because the second row is a multiple of the first row.
$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ -4 & -5 & -6 \end{bmatrix}$ is not invertible because it contains a zero row (and also you can check that its determinant is 0).

How to find the inverse of a matrix (if it exists). Put the given matrix and the identity matrix of the same size in one "fat" matrix as shown in the example below. Reduce the whole fat matrix to RREF. If you get the identity matrix on the left, then the matrix on the right is the inverse of the original matrix. If you get some zero rows on the left, then the original matrix is not invertible.
It can be shown that the inverse of a 2×2 matrix can be found using the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Example 16. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -11 & -7 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{11}{3} & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 6 & -3 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{11}{3} & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & -\frac{4}{3} & 1 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{11}{3} & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

Therefore $A^{-1} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Exercises

1. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 6 & -7 \end{bmatrix}$, and $C = \begin{bmatrix} 4 & -3 & 0 \\ 0 & 2 & 1 \end{bmatrix}$. Find the following if defined.
   (a) $A + B$ and $A + C$
   (b) $3A$
   (c) $-B$
   (d) $2A + 3B$
   (e) AB, AC, and CB

2. Find the determinant of
   (a) $A = \begin{bmatrix} 4 & 1 \\ 7 & 3 \end{bmatrix}$
   (b) $B = \begin{bmatrix} -6 & 0 & 0 \\ 5 & -4 & 0 \\ 1 & -3 & 2 \end{bmatrix}$
   (c) $C = \begin{bmatrix} -1 & 2 & 5 \\ 1 & 0 & -2 \\ 2 & -1 & 3 \end{bmatrix}$

3. Find the inverse of
   (a) $A = \begin{bmatrix} 3 & 1 \\ 7 & 3 \end{bmatrix}$
   (b) $B = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$
   (c) $C = \begin{bmatrix} -1 & 2 & 5 \\ 1 & -1 & -8 \\ 2 & -8 & 3 \end{bmatrix}$

4. Give an example of matrices A and B such that $\det(A) + \det(B) \neq \det(A + B)$.

5. Give an example of nonzero matrices A and B (of any size you like) such that $AB = 0$.

6. Prove that if a matrix is invertible, then its inverse is unique.

7. Prove that the sum, difference, and product of two diagonal matrices are diagonal. Also prove that the inverse of a diagonal matrix (if it exists) is diagonal.

8. Prove that the product of two upper triangular matrices is upper triangular.

9. Prove that the inverse of an upper triangular matrix (if it exists) is upper triangular.

Selected Answers

1. (a) $\begin{bmatrix} 6 & -2 \\ 9 & -3 \end{bmatrix}$; undefined

   (b) $\begin{bmatrix} 3 & -6 \\ 9 & 12 \end{bmatrix}$

   (c) $\begin{bmatrix} -5 & 0 \\ -6 & 7 \end{bmatrix}$

2. (a) 5

   (b) 48

3. (c) $\begin{bmatrix} 67 & 46 & 11 \\ 19 & 13 & 3 \\ 6 & 4 & 1 \end{bmatrix}$