Math 151

Practice problems for Test 1

The actual test will consist of 5 problems. You will have 50 minutes to complete the test.

1. Let $a = 51$ and $b = 85$.
   (a) Find the greatest common divisor d of a and b.
   (b) Find $m, n \in \mathbb{Z}$ such that $d = am + bn$.
2. Prove that if a and b are integers, then $b|a \;\Leftrightarrow\; a\mathbb{Z} \subset b\mathbb{Z}$.
3. Prove that there are infinitely many prime numbers.
4. State the Fundamental Theorem of Arithmetic.
5. Prove that if an integer n is divisible by 2 and 3, then it is divisible by 6.
6. Solve the congruences:
   (a) $15x \equiv 21 (\text{mod } 24)$
   (b) $15x \equiv 8 (\text{mod } 24)$
   i.e. find all distinct solutions (and explain why there are no other solutions).
7. Solve the system $x \equiv 6 (\text{mod } 25)$, $x \equiv 2 (\text{mod } 11)$.
8. (a) Give the definition of $\phi(n)$.
   (b) Find $\phi(15)$.
   (c) Prove that if p and q are distinct primes, then $\phi(pq) = pq - p - q + 1$.
9. Find the multiplicative order and inverse of $[101]$ in $\mathbb{Z}_{1000}$.
10. Which of the following are well-defined functions?
    (a) $f : \mathbb{Z}_8 \to \mathbb{Z}_{12}$ given by $f([x]_8) = [3x]_{12}$
    (b) $f : \mathbb{Z}_8 \to \mathbb{Z}_{16}$ given by $f([x]_8) = [-x]_{16}$
    (c) $f : \mathbb{Z}_{16} \to \mathbb{Z}_{16}$ given by $f([x]_8) = [-x]_{16}$
11. For each function in the previous problem, determine
    (a) whether it is one-to-one,
    (b) whether it is onto,
    (c) the image,
    (d) the factor set.
12. Define $f : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $f([x]_{mn}) = ([x]_m, [x]_n)$. Show that f is a well-defined function.

13. Consider the set of real numbers $\mathbb{R}$. Which of the following are equivalence relations? For those that are, show that all the conditions for an equivalence relation (reflexive, symmetric, and transitive laws) are satisfied, and describe the equivalence classes. For those that are not, say which condition(s) is (are) not satisfied, and why (give a specific example to show that a condition is not satisfied).

    (a) $x \sim y$ if $|x - y| \leq 1$

    (b) $x \sim y$ if $|x| = |y|$

    (c) $x \sim y$ if $xy > 0$

    (d) $x \sim y$ if either $xy > 0$ or $x = y = 0$.

14. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 4 & 5 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 2 & 3 & 6 \end{pmatrix}$.

    (a) Compute $\sigma\tau$ and $\tau\sigma$.

    (b) Do σ and τ commute?

    (c) Find σ^{-1} and τ^{-1}.

    (d) Write σ and τ as products of disjoint cycles.

    (e) Draw the associated diagrams for σ and τ.

    (f) Write σ as a product of transpositions.

    (g) Are σ and τ even or odd permutations?