Math 151

Fall 2008

Test 2 - Solutions

1. (a) Find the order of each group, whether it is abelian, and whether it is cyclic. Provide brief explanations (you may refer to a theorem or an example in the book).

group	order	abelian?	cyclic?
$\mathbb{Z}_{12}$	12 $\lvert\mathbb{Z}_n\rvert = n$	*yes* *by example 3.1.3*	*yes* *by example 3.2.8*
$\mathbb{Z}_3 \times \mathbb{Z}_4$	12 $3 \times 4 = 12$	*yes* *both* $\mathbb{Z}_3$ *and* $\mathbb{Z}_4$ *are abelian*	*yes* *isomorphic to* $\mathbb{Z}_{12}$ *by theorem 3.5.5 (or because* $(1, 1)$ *is a generator)*
$\mathbb{Z}_2 \times \mathbb{Z}_6$	12 $2 \times 6 = 12$	*yes* *both* $\mathbb{Z}_2$ *and* $\mathbb{Z}_6$ *are abelian*	*no* *contains no element of order* 12
$\mathbb{Z}_{12}^\times$	4 *elements: 1, 5, 7, 11 (or* $\lvert\mathbb{Z}_{12}^\times\rvert = \phi(12) = 4$*)*	*yes* *by example 3.1.4*	*no* *contains no element of order* 4
S_{12}	12! $\lvert S_n\rvert = n!$	*no* *e.g.* $(123)(12) \neq (12)(123)$	*no* *every cyclic group is abelian*
A_4	12 $\lvert A_n\rvert = n!/2$ *(or by problem 13 in section 3.6)*	*no* *e.g.* $(123)(124) \neq (124)(123)$	*no* *every cyclic group is abelian*
$GL_{12}(\mathbb{R})$	∞ *for each* $x \in \mathbb{R}$, $xI_{12} \in GL_{12}(\mathbb{R})$	*no* *matrix multiplication is not commutative*	*no* *every cyclic group is abelian*

(b) Are any of the above groups isomorphic? (Explain.)
Only groups that have the same order and same properties (either both abelian or both nonabelian; either both cyclic or both noncyclic) can be isomorphic. Thus among the above groups, only $\mathbb{Z}_{12}$ *and* $\mathbb{Z}_3 \times \mathbb{Z}_4$ *be isomorphic. This two groups are indeed isomorphic by Proposition 3.4.5.*

2. Let $\phi(\mathbb{Z}_2 \times \mathbb{Z}_4) \to \mathbb{Z}_2$ be defined by $\phi([a]_2, [b]_4) = [a + b]_2$.

(a) Show that ϕ is a homomorphism.
Let $([a]_2, [b]_4), ([c]_2, [d]_4) \in \mathbb{Z}_2 \times \mathbb{Z}_4$. *Then* $\phi(([a]_2, [b]_4) + ([c]_2, [d]_4)) = \phi([a + c]_2, [b + d]_4) = [(a + c) + (b + d)]_2 = [(a + b) + (c + d)]_2 = [a + b]_2 + [c + d]_2 = \phi([a]_2, [b]_4) + \phi([c]_2, [d]_4)$.

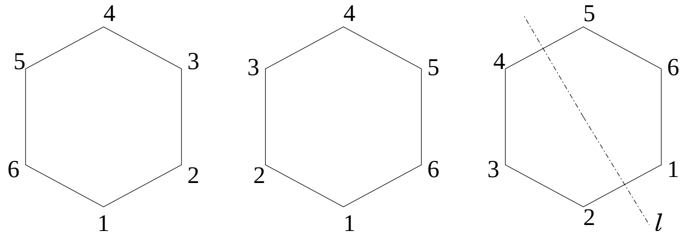
(b) Find the kernel of ϕ.

$ker(\phi) = \{([a]_2, [b]_4) \in \mathbb{Z}_2 \times \mathbb{Z}_4 \mid [a+b]_2 = [0]_2\} = \{([a]_2, [b]_4) \in \mathbb{Z}_2 \times \mathbb{Z}_4 \mid a + b \equiv 0 \ (mod\ 2)\} = \{([0]_2, [0]_4), ([0]_2, [2]_4), ([1]_2, [1]_4), ([1]_2, [3]_4)\}$.

3. Prove that $\mathbb{Z}$ is a normal subgroup of $\mathbb{R}$.
   For any $a \in \mathbb{Z}$ and $b \in \mathbb{R}$, $b + a + (-b) = a \in \mathbb{Z}$, thus $\mathbb{Z}$ is normal.

4. In D_6, let a and b denote the counterclockwise rotation through an angle of 60 degrees and the flip about the vertical line, respectively.

   (a) Give a geometric description of the rigid motion ab. (Is it a rotation? If so, through what angle? Or is it a flip? If so, about what line? Or is it some other rigid motion?)
   By performing first b and then a as shown below, we see that ab is the flip about line l.



   (b) What is the order of ab?
   The order of ab is 2 (since the order of any flip is 2).

5. Let G be any group. Define a function $\phi : G \to G$ by $\phi(x) = x^{-1}$ for all $x \in G$.

   (a) Prove that ϕ is one-to-one and onto.
   If $\phi(a) = \phi(b)$, i.e. $a^{-1} = b^{-1}$, then $aa^{-1}b = ab^{-1}b$. Therefore $b = a$. Thus ϕ is one-to-one.
   For any $y \in G$, let $x = y^{-1}$. Then $\phi(x) = x^{-1} = (y^{-1})^{-1} = y$. Thus ϕ is onto.

   (b) Give an example of a group G for which the function ϕ defined above is an isomorphism (and prove that it is).
   Let $G = \mathbb{R}^{\times}$. Then $\phi(ab) = (ab)^{-1} = a^{-1}b^{-1} = \phi(a)\phi(b)$. Since ϕ is also one-to-one and onto as shown above, ϕ is an isomorphism.

Optional: Prove that any group of order 24 contains at least one element of order 2.

Let G be a group of order 24. Possible orders of elements of G are divisors of 24: 1, 2, 3, 4, 6, 8, 12, and 24. The only element of order 1 is the identity element. If there is an element of an even order, say a of order $2k$ (where $k \in \mathbb{Z}$), then a^k has order 2 since $(a^k)^2 = a^{2k} = e$. If there is no element of any even order, then all elements except e must have order 3. However, we will show that this case is impossible. For each element a of order 3, consider the pair $\{a, a^2\}$ (note that $a \neq a^2$). We will show that such pairs are disjoint. Indeed, if for some elements a and b we have $a = b$, then $a^2 = b^2$, so $\{a, a^2\} = \{b, b^2\}$; if $a = b^2$, then $a^2 = b^4 = b^3b = eb = b$, so $\{a, a^2\} = \{b, b^2\}$; if $a^2 = b$, then $a = ea = a^3a = a^4 = (a^2)^2 = b^2$, and $\{a, a^2\} = \{b, b^2\}$; finally, if $a^2 = b^2$, then $a = a^4 = b^4 = b$, so again $\{a, a^2\} = \{b, b^2\}$. But 23 elements (i.e. all but the identity element) cannot form disjoint pairs. We have a contradiction. Thus there must be at least one element of an even order.