Math 171 Solutions to homework problems Spring 2005

Section 1.1

2(c) Solve $|x^2 - x - 1| < x^2$.

Since $x^2 \geq 0$ for all x, by Theorem 1.6 the given inequality is equivalent to
$-x^2 < x^2 - x - 1 < x^2$, that is,
$-x^2 < x^2 - x - 1$ and $x^2 - x - 1 < x^2$. So we solve both inequalities and take the intersection of their solution sets.

First we solve $-x^2 < x^2 - x - 1$.
$2x^2 - x - 1 > 0$
$(2x + 1)(x - 1) > 0$
Case 1: $2x + 1 > 0$ and $x - 1 > 0$, so $x > -\frac{1}{2}$ and $x > 1$, therefore $x > 1$.
Case 2: $2x + 1 < 0$ and $x - 1 < 0$, so $x < -\frac{1}{2}$ and $x < 1$, therefore $x < -\frac{1}{2}$.
Answer: $(-\infty, -\frac{1}{2}) \cup (1, \infty)$.

Now we solve $x^2 - x - 1 < x^2$.
$x + 1 > 0$, so $x > -1$.
Answer: $(-1, \infty)$.

The intersection of the two answers above is $(-1, -\frac{1}{2}) \cup (1, \infty)$.

4(7) Prove that $0 \leq a < b$ and $0 \leq c < d$ imply $ac < bd$. Show that this statement is false if the hypothesis $a \geq 0$ is removed.

Since $c \geq 0$, multiplying both sides of $a < b$ gives $ac \leq bc$.
Since $b > 0$, multiplying both sides of $c < d$ gives $bc < bd$.
So $ac \leq bc < bd$. By transitivity, we have $ac < bd$.

If the hypothesis $a \geq 0$ is removed and we only have $a < b$ and $0 \leq c < d$ then $ac < bd$ may not hold.
Counterexample: $a = -2$, $b = -1$, $c = 3$, $d = 7$. Then $ac = -6$ and $bd = -7$.

7(a) Prove that $|x| < 1$ implies $|x^2 - 1| \leq 2|x - 1|$.

The inequality $|x| \leq 1$ implies that $-1 \leq x \leq 1$ (we use Theorem 1.6 again). Adding 1 gives $0 \leq x + 1 \leq 2$, and since $-2 < 0$ we have $-2 \leq x + 1 \leq 2$. Therefore $|x + 1| \leq 2$. (Another way to get this is to use the triangle inequality: $|x + 1| \leq |x| + |1| \leq 1 + 1 = 2$.)

Now multiply both sides of the inequality $|x + 1| \leq 2$ by $|x - 1|$ (we can do this because $|x - 1| \geq 0$):
$|x + 1| \cdot |x - 1| \leq 2|x - 1|$.
By the multiplicative property of the absolute value, $|(x + 1)(x - 1)| \leq 2|x - 1|$.
So $|x^2 - 1| \leq 2|x - 1|$.

8(a) Find all values of $n \in \mathbb{N}$ that satisfy the inequality $\dfrac{1-n}{1-n^2} < 0.01$.

Rewrite the inequality as $\dfrac{1-n}{(1-n)(1+n)} < \dfrac{1}{100}$ and simplify (note that $n \neq 1$):
$\dfrac{1}{1+n} < \dfrac{1}{100}$
$100 < 1 + n$ (since both 100 and $1 + n$ are positive)
$n > 99$.

Section 1.2

1(c) Prove that the formula $\sum_{k=1}^{n} \frac{a-1}{a^k} = 1 - \frac{1}{a^n}$ **holds for all** $n \in \mathbb{N}$ **and** $a \neq 0$**.**

We will prove this formula by induction on n.

Basis step: if $n = 1$, the formula says $\sum_{k=1}^{1} \frac{a-1}{a^k} = 1 - \frac{1}{a^1}$. The sum on the left hand side has only one term ($k = 1$), so the left hand side is $\frac{a-1}{a^1} = \frac{a}{a} - \frac{1}{a} = 1 - \frac{1}{a}$.

Inductive step: assume that the formula holds for $n = m$ (althought traditionally k is used in this step instead of m, in our problem the letter k is reserved for the summation index, so we have to use something else here), so assume that $\sum_{k=1}^{m} \frac{a-1}{a^k} = 1 - \frac{1}{a^m}$ is true. We want to prove that the formula is true for $n = m + 1$, i.e. $\sum_{k=1}^{m+1} \frac{a-1}{a^k} = 1 - \frac{1}{a^{m+1}}$.

Using the inductive hypothesis, we have: $\sum_{k=1}^{m+1} \frac{a-1}{a^k} = \sum_{k=1}^{m} \frac{a-1}{a^k} + \frac{a-1}{a^{m+1}} = 1 - \frac{1}{a^m} + \frac{a-1}{a^{m+1}} = 1 - \frac{1}{a^m} + \frac{a}{a^{m+1}} - \frac{1}{a^{m+1}} = 1 - \frac{1}{a^m} + \frac{1}{a^m} - \frac{1}{a^{m+1}} = 1 - \frac{1}{a^{m+1}}$. This completes the proof.

2(a) Use the Binomial Formula to prove that $2^n = \sum_{k=0}^{n} \binom{n}{k}$ **for all** $n \in \mathbb{N}$**.**

The Binomial Formula says that for any $a, b \in \mathbb{R}$ and any $n \in \mathbb{N}$, $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

In particular, if $a = b = 1$, it gives $(1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k$, or $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

6(a) Prove that the inequality $n < 2^n$ **holds for all** $n \in \mathbb{N}$**.**

The proof is by induction on n.
Basis step: if $n = 1$, $1 < 2^1$ is true.
Inductive step: assume that the inequality holds for $n = k$, i.e. $k < 2^k$. We want to prove that the inequality holds for $n = k + 1$, i.e. $k + 1 < 2^{k+1}$.
We have: $k + 1 < 2^k + 1 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$.

6(b) Prove that the inequality $n^2 \leq 2^n + 1$ **holds for all** $n \in \mathbb{N}$**.**

Using the hint in the book, we first prove that the inequality $2n + 1 < 2^n$ holds for $n \geq 3$.
Basis step: if $n = 3$, $2 \cdot 3 + 1 < 2^3$ says that $7 < 8$ which is true.
Inductive step: assume $2k + 1 < 2^k$ is true, we want to prove that $2(k+1) + 1 < 2^{k+1}$ is true.
We have: $2(k + 1) + 1 = 2k + 3 = (2k + 1) + 2 < 2^k + 2 < 2^k + 2^k = 2^{k+1}$.

Now we are ready to prove the inequality $n^2 \leq 2^n + 1$ for $n \geq 3$.
Basis step: if $n = 3$, $3^2 \leq 2^3 + 1$ is true since $9 = 8 + 1$.
Inductive step: assume that $k^2 \leq 2^k + 1$ is true, we want to prove that $(k + 1)^2 \leq 2^{k+1} + 1$ is true (where $k \geq 3$).
Using the inductive hypothesis and the inequality proved above we have:
$(k + 1)^2 = k^2 + (2k + 1) \leq 2^k + 1 + 2^k = 2 \cdot 2^k + 1 = 2^{k+1} + 1$.

Finally, we have to check our inequality for $n = 1$ and $n = 2$ since we only proved if for $n \geq 3$.
If $n = 1$, the inequality is $1^2 \leq 2^1 + 1$ which is true, and if $n = 2$, it is $2^2 \leq 2^2 + 1$ which is also true. This completes the proof.