Section 5.1

8(a) If $f$ is increasing on $[a, b]$ and $P = \{x_0, \ldots, x_n\}$ is any partition of $[a, b]$, prove that 
\[
\sum_{j=1}^{n} (M_j(f) - m_j(f))(x_j - x_{j-1}) \leq (f(b) - f(a))||P||.
\]

We need two observations. (1) For each $j$, $x_j - x_{j-1} \leq ||P||$.
(2) For each $j$, $M_j(f) = m_{j+1}(f)$ since $f$ is increasing.

Then \[
\sum_{j=1}^{n} (M_j(f) - m_{j+1}(f))(x_j - x_{j-1}) \leq \sum_{j=1}^{n} (M_j(f) - m_j(f))||P|| = \left( \sum_{j=1}^{n} (M_j(f) - m_j(f)) \right) ||P||
\]
\[
= (M_n(f) - m_n(f) + M_{n-1}(f) - m_{n-1}(f) + \ldots + M_2(f) - m_2(f) + M_1(f) - m_1(f))||P||
\]
\[
= (M_n(f) - m_1(f))||P|| = (f(b) - f(a))||P||.
\]

8(b) Prove that if $f$ is monotone on $[a, b]$, then $f$ is integrable on $[a, b]$.

Case I. $f$ is increasing.

Since $f$ is increasing, it is bounded on $[a, b]$. Let $\varepsilon > 0$ be given, then choose a partition $P$ such that $||P|| \leq \frac{\varepsilon}{f(b) - f(a)}$ (if $f(b) - f(a) = 0$, choose any partition $P$). Then
\[
U(f, P) - L(f, P) = \sum_{j=1}^{n} (M_j(f) - m_j(f))(x_j - x_{j-1}) \leq (f(b) - f(a))||P|| < \varepsilon.
\]

Case II. $f$ is decreasing.

Then $-f$ is increasing. By above, $-f$ is integrable, therefore $f = -(-f)$ is integrable.

9 Let $f$ be bounded on a nondegenerate interval $[a, b]$. Prove that $f$ is integrable on $[a, b]$ if and only if given $\varepsilon > 0$ there is a partition $P_{\varepsilon}$ of $[a, b]$ such that $P \supset P_{\varepsilon}$ implies $|U(f, P) - L(f, P)| < \varepsilon$.

($\Rightarrow$) If $f$ is integrable then for any $\varepsilon > 0$ there exists a partition $P_{\varepsilon}$ such that $U(f, P_{\varepsilon}) - L(f, P_{\varepsilon}) < \varepsilon$. Then for any refinement $P$ of $P_{\varepsilon}$ we have $L(f, P_{\varepsilon}) \leq L(f, P) \leq U(f, P) \leq U(f, P_{\varepsilon})$, therefore $|U(f, P) - L(f, P)| \leq |U(f, P_{\varepsilon}) - L(f, P_{\varepsilon})| < \varepsilon$.

($\Leftarrow$) If for any $\varepsilon > 0$ there is a partition $P_{\varepsilon}$ of $[a, b]$ such that $P \supset P_{\varepsilon}$ implies $|U(f, P) - L(f, P)| < \varepsilon$, then since $P_{\varepsilon}$ is a refinement of itself, we have $|U(f, P_{\varepsilon}) - L(f, P_{\varepsilon})| < \varepsilon$, therefore $f$ is integrable.

Section 5.2

1 Using the connection between integrals and area, evaluate each of the following integrals.

(a) $\int_{0}^{1} |x - 0.5| \, dx = A_1 + A_2 = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
(b) \( a > 0 \) \( \int_0^a \sqrt{a^2 - x^2} \, dx = A = \frac{1}{4} \pi a^2 \)

(c) \( \int_{-2}^2 (|x+1| + |x|) \, dx = A_1 + A_2 + A_3 = \frac{1}{2} (3+1) \cdot 1 + 1 + \frac{1}{2} (1+5) \cdot 2 = 2 + 1 + 6 = 9 \)

(d) \( a < b \) \( \int_a^b (3x + 1) \, dx \)

Case I. \( a \geq -\frac{1}{3} \)

\[ \int_a^b (3x + 1) \, dx = A = \frac{1}{2} (3a + 1 + 3b + 1)(b - a) = \frac{3}{2} (b^2 - a^2) + (b - a) \]
Case II. $b \leq -\frac{1}{3}$.

\[ \int_a^b (3x + 1)\,dx = -A = \frac{1}{2}(-3a - 1 - 3b - 1)(b - a) = \frac{3}{2}(b^2 - a^2) + (b - a) \]

Case III. $a < \frac{1}{3} < b$.

\[ \int_a^b (3x + 1)\,dx = A_1 - A_2 = \frac{1}{2}(3b + 1) \left( b + \frac{1}{3} \right) - \frac{1}{2}(-3a - 1) \left( -\frac{1}{3} - a \right) = \frac{3}{2}(b^2 - a^2) + (b - a) \]

4 Suppose that $a < b < c$ and $f$ is integrable on $[a, c]$. Prove that

\[ \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \]

By theorem 5.20, \[ \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx, \] therefore

\[ \int_a^b f(x)\,dx = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \]