Math 171 Solutions to homework problems Spring 2005

Section 5.1

8(a) If f is increasing on $[a,b]$ and $P = \{x_0, \dots, x_n\}$ is any partition of $[a,b]$, prove that $\sum_{j=1}^{n}(M_j(f) - m_j(f))(x_j - x_{j-1}) \le (f(b) - f(a))||P||$.

We need two observations. (1) For each j, $x_j - x_{j-1} \le ||P||$.
(2) For each j, $M_j(f) = m_{j+1}(f)$ since f is increasing.

Then $\sum_{j=1}^{n}(M_j(f)-m_j(f))(x_j-x_{j-1}) \le \sum_{j=1}^{n}(M_j(f)-m_j(f))||P|| = \left(\sum_{j=1}^{n}(M_j(f) - m_j(f))\right)||P||$
$= (M_n(f) - m_n(f) + M_{n-1}(f) - m_{n-1}(f) + \ldots + M_2(f) - m_2(f) + M_1(f) - m_1(f))||P||$
$= (M_n(f) - m_1(f))||P|| = (f(b) - f(a))||P||.$

8(b) Prove that if f is mononotone on $[a,b]$, then f is integrable on $[a,b]$.

Case I. f is increasing.
Since f is increasing, it is bounded on $[a,b]$. Let $\varepsilon > 0$ be given, then choose a partition P such that $||P|| \le \frac{\varepsilon}{f(b) - f(a)}$ (if $f(b) - f(a) = 0$, choose any partition P). Then
$U(f,P) - L(f,P) = \sum_{j=1}^{n}(M_j(f) - m_j(f))(x_j - x_{j-1}) \le (f(b) - f(a))||P|| < \varepsilon.$

Case II. f is decreasing.
Then $-f$ is increasing. By above, $-f$ is integrable, therefore $f = -(-f)$ is integrable.

9 Let f be bounded on a nondegenerate interval $[a,b]$. Prove that f is integrable on $[a,b]$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a,b]$ such that $P \supset P_\varepsilon$ implies $|U(f,P) - L(f,P)| < \varepsilon$.

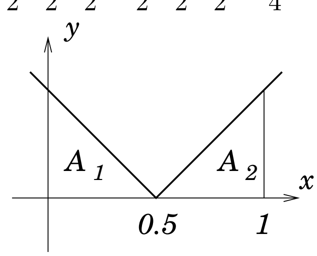
($\Rightarrow$) *If f is integrable then for any $\varepsilon > 0$ there exists a partition P_ε such that $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$. Then for any refinement P of P_ε we have $L(f,P_\varepsilon) \le L(f,P) \le U(f,P) \le U(f,P_\varepsilon)$, therefore $|U(f,P) - L(f,P)| \le |U(f,P_\varepsilon) - L(f,P_\varepsilon)| < \varepsilon$.*

($\Leftarrow$) *If for any $\varepsilon > 0$ there is a partition P_ε of $[a,b]$ such that $P \supset P_\varepsilon$ implies $|U(f,P) - L(f,P)| < \varepsilon$, then since P_ε is a refinement of itself, we have $|U(f,P_\varepsilon) - L(f,P_\varepsilon)| < \varepsilon$, therefore f is integrable.*

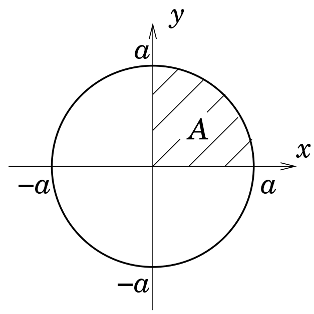
Section 5.2

1 Using the connection between integrals and area, evaluate each of the following integrals.

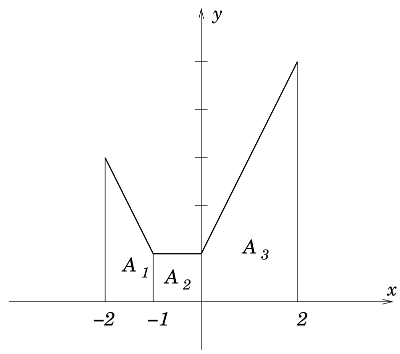
(a) $\int_0^1 |x - 0.5|dx = A_1 + A_2 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$



(b) $(a > 0)$ $\int_0^a \sqrt{a^2 - x^2}dx = A = \frac{1}{4}\pi a^2$

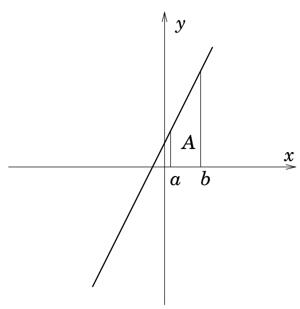


(c) $\int_{-2}^{2}(|x+1| + |x|)dx = A_1 + A_2 + A_3 = \frac{1}{2}(3+1)\cdot 1 + 1 + \frac{1}{2}(1+5)\cdot 2 = 2 + 1 + 6 = 9$



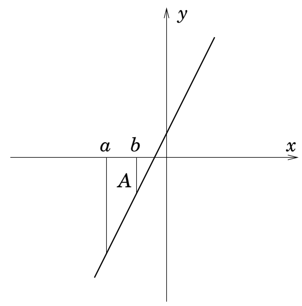
(d) $(a < b)$ $\int_a^b (3x+1)dx$

Case I. $a \geq -\frac{1}{3}$.



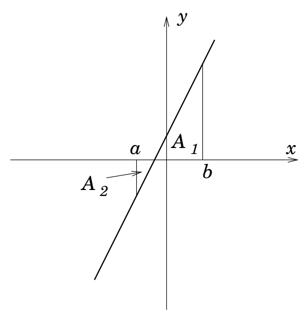
$\int_a^b (3x+1)dx = A = \frac{1}{2}(3a + 1 + 3b + 1)(b - a) = \frac{3}{2}(b^2 - a^2) + (b - a)$

Case II. $b \leq -\frac{1}{3}$.



$$\int_a^b (3x+1)dx = -A = \frac{1}{2}(-3a-1-3b-1)(b-a) = \frac{3}{2}(b^2-a^2)+(b-a)$$

Case III. $a < -\frac{1}{3} < b$.



$$\int_a^b (3x+1)dx = A_1 - A_2 = \frac{1}{2}(3b+1)\left(b+\frac{1}{3}\right) - \frac{1}{2}(-3a-1)\left(-\frac{1}{3}-a\right) = \frac{3}{2}(b^2-a^2)+(b-a)$$

4 Suppose that $a < b < c$ and f is integrable on $[a, c]$. Prove that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

By theorem 5.20, $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$, *therefore* $\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.