Math 171 Solutions to homework problems Spring 2005

Section 5.3

1 Compute each of the following integrals.

(a) $\int_{-3}^{3} |x^2 + x - 2| dx$

Since $x^2 + x - 2 = (x+2)(x-1)$ *is nonnegative on* $[-3,-2]$ *and on* $[1,3]$ *and nonpositive on* $[-2,1]$,

$|x^2 + x - 2| = \begin{cases} x^2 + x - 2 & \text{if } x \in [1,3] \cup [-2,1] \\ -x^2 - x + 2 & \text{if } x \in [-2,1] \end{cases}$, *so*

$$\int_{-3}^{3} |x^2+x-2|dx = \int_{-3}^{-2} |x^2+x-2|dx + \int_{-2}^{1} |x^2+x-2|dx + \int_{1}^{3} |x^2+x-2|dx$$

$$= \int_{-3}^{-2} (x^2+x-2)dx + \int_{-2}^{1} (-x^2-x+2)dx + \int_{1}^{3} (x^2+x-2)dx$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{2} - 2x\right)\Big|_{-3}^{-2} + \left(-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right)\Big|_{-2}^{1} + \left(\frac{x^3}{3} + \frac{x^2}{2} - 2x\right)\Big|_{1}^{3} = 15$$

(b) $\int_1^4 \frac{\sqrt{x}-1}{\sqrt{x}} dx = \int_1^4 \left(1 - \frac{1}{\sqrt{x}}\right) dx = \left(x - 2\sqrt{x}\right)\big|_1^4 = 0 - (-1) = 1$

(c) $\int_0^1 (3x+1)^{99} dx = \int_1^4 u^{99} \frac{1}{3} du = \frac{u^{100}}{300}\Big|_1^4 = \frac{4^{100}-1}{300}$ *(substitution:* $u = 3x+1$, $\frac{1}{3}du = dx$*)*

(d) $\int_1^e x \log x dx = \frac{x^2 \log x}{2}\Big|_1^e - \int_1^e \frac{x^2}{2x} dx = \left(\frac{e^2}{2} - 0\right) - \frac{x^2}{4}\Big|_1^e = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}$

(itegration by parts: $u = \log x$, $du = \frac{1}{x}dx$, $dv = xdx$, $v = \frac{x^2}{2}$*.)*

(e) $\int_0^{\pi/2} e^x \sin x dx$ *(integration by parts:* $u = e^x$, $du = e^x dx$, $dv = \sin x dx$, $v = -\cos x$*.)*

$$= -e^x \cos x|_0^{\pi/2} + \int_0^{\pi/2} e^x \cos x dx$$

(integration by parts again: $u = e^x$, $du = e^x dx$, $dv = \cos x dx$, $v = \sin x$*.)*

$$= (0 - (-1)) + e^x \sin x|_0^{\pi/2} - \int_0^{\pi/2} e^x \sin x dx$$

$$2\int_0^{\pi/2} e^x \sin x dx = 1 + (e^{\pi/2} - 0) \qquad \Rightarrow \qquad \int_0^{\pi/2} e^x \sin x dx = \frac{e^{\pi/2}+1}{2}$$

3(a) If $f : [0, +\infty) \to \mathbb{R}$ is continuous, find $\frac{d}{dx}\int_1^{x^2} f(t)dt$.

Let $F(x)$ *be an antiderivative of* $f(x)$, *i.e.* $F'(x) = f(x)$. *Then* $\int_1^{x^2} f(t)dt = F(x^2) - F(1)$, *so* $\frac{d}{dx}\int_1^{x^2} f(t)dt = (F(x^2) - F(1))' = F'(x^2) \cdot 2x - 0 = f(x^2) \cdot 2x$.

3(b) If $h : \mathbb{R} \to \mathbb{R}$ is continuous, find $\dfrac{d}{dt}\displaystyle\int_{\cos t}^{t} h(x)dx$.

Let $H(t)$ be an antiderivative of $h(t)$, i.e. $H'(t) = h(t)$. Then $\displaystyle\int_{\cos t}^{t} h(x)dx = H(t) - H(\cos t)$, so $\dfrac{d}{dt}\displaystyle\int_{\cos t}^{t} h(x)dx = (H(t) - H(\cos t))' = H'(t) + H'(\cos t)\sin t = h(t) + h(\cos t)\sin t$.

4 Define $L : (0, \infty) \to \mathbb{R}$ by $L(x) = \displaystyle\int_1^x \frac{dt}{t}$.

(a) Prove that L is differentiable and strictly increasing on $(0, \infty)$, with $L'(x) = \dfrac{1}{x}$ and $L(1) = 0$.

Since $L(x) = \displaystyle\int_1^x \frac{1}{t}dt$, by the Fundamental theorem of Calculus, L is differentiable, and $L'(x) = \dfrac{1}{x}$.

Since $L'(x) > 0$ on $(0, \infty)$, $L(x)$ is strictly increasing on $(0, \infty)$.

$L(1) = \displaystyle\int_1^1 \frac{dt}{t} = 0.$

(c) Using the fact that $(x^q)' = qx^{q-1}$ for $x > 0$ and $q \in \mathbb{Q}$, prove that $L(x^q) = qL(x)$ for all $q \in \mathbb{Q}$ and $x > 0$. (Hint: compare derivatives of $L(x^q)$ and $qL(x)$.)

$(L(x^q))' = L'(x^q) \cdot (x^q)' = \dfrac{1}{x^q} \cdot qx^{q-1} = \dfrac{q}{x} = q \cdot \dfrac{1}{x} = (qL(x))'.$

Then $L(x^q) = qL(x) + c$.

$L(1) = qL(1) + c.$

Since $L(1) = 0$, we have $c = 0$.

So $L(x^q) = qL(x)$.